THE RELIABILITY OF DATA IN PAIRWISE COMPARISON MATRICES

Jacek Szybowski

Faculty of Applied Mathematics
AGH University of Science and Technology
Poland
Supported by the National Science Centre, Poland
as a part of the project No. 2017/25/B/HS4/01617

Hong Kong, July 13th, 2018

Introductory definitions

$E_1, \ldots, E_n$ - alternatives to be compared
$\mathbf{A} = (a_{ij}) \wedge a_{ij} \in \mathbb{R}_+ \wedge i, j \in \{1, \ldots, n\}$ – *pairwise comparisons matrix* (PC-matrix).
$a_{ij} \simeq \frac{E_i}{E_j}$

- $\mathbf{A}$ is *reciprocal* if for all $i, j \in \{1, \ldots, n\} : a_{ij} = \frac{1}{a_{ji}}$.
- $\mathbf{A}$ is *consistent* if for all $i, j, k \in \{1, \ldots, n\} : a_{ij} \cdot a_{jk} = m_{ik}$.
- For $i < j < k$ each triple $\{a_{ij}, a_{jk}, a_{ik}\}$ is called a *triad*.

Introductory definitions

$E_1, \ldots, E_n$ - alternatives to be compared
$\mathbf{A} = (a_{ij}) \wedge a_{ij} \in \mathbb{R}_+ \wedge i,j \in \{1,\ldots,n\}$ – *pairwise comparisons matrix* (PC-matrix).
$a_{ij} \simeq \frac{E_i}{E_j}$

- $\mathbf{A}$ is *reciprocal* if for all $i,j \in \{1,\ldots,n\} : a_{ij} = \frac{1}{a_{ji}}$.
- $\mathbf{A}$ is *consistent* if for all $i,j,k \in \{1,\ldots,n\} : a_{ij} \cdot a_{jk} = m_{ik}$.
- For $i < j < k$ each triple $\{a_{ij}, a_{jk}, a_{ik}\}$ is called a *triad*.

Eigenvalue Method

A – a reciprocal square matrix
The Eigenvalue Method (EV) (Saaty 1977):
Find $\lambda > 0$ and $w = (w_1, \ldots, w_n)$ with positive coordinates such that

$$\mathbf{A}w = \lambda w$$

and $(\exists v = (v_1, \ldots, v_n) \neq 0 \ \mathbf{A}v = \mu v) \Rightarrow |\mu| \leq |\lambda|$.
Number λ is *an eigenvalue*. Vector w is called *a principal right eigenvector* and it serves as a priority vector.

Eigenvalue Method

A – a reciprocal square matrix
The Eigenvalue Method (EV) (Saaty 1977):
Find $\lambda > 0$ and $w = (w_1, \ldots, w_n)$ with positive coordinates such that

$$\mathbf{A}w = \lambda w$$

and $(\exists v = (v_1, \ldots, v_n) \neq 0 \ \mathbf{A}v = \mu v) \Rightarrow |\mu| \leq |\lambda|$.
Number λ is *an eigenvalue*. Vector w is called *a principal right eigenvector* and it serves as a priority vector.

Consistency Index

Basing on this method, Saaty introduced the Consistency Index:

$$CI(\mathbf{A}) = \frac{\lambda - n}{n-1}.$$

REMARK

$CI(\mathbf{A}) = 0 \Leftrightarrow \mathbf{A}$ is consistent.

THEOREM (Aguarón and Moreno-Jiménez, 2003)

$$CI(\mathbf{A}) = \frac{1}{n(n-1)} \sum_{i<j} e_{ij},$$

where

$$e_{ij} = a_{ij}\frac{w_j}{w_i} + a_{ji}\frac{w_i}{w_j} - 2.$$

Consistency Index

Basing on this method, Saaty introduced the Consistency Index:

$$CI(\mathbf{A}) = \frac{\lambda - n}{n - 1}.$$

REMARK

$CI(\mathbf{A}) = 0 \Leftrightarrow \mathbf{A}$ is consistent.

THEOREM (Aguarón and Moreno-Jiménez, 2003)

$$CI(\mathbf{A}) = \frac{1}{n(n-1)} \sum_{i<j} e_{ij},$$

where

$$e_{ij} = a_{ij}\frac{w_j}{w_i} + a_{ji}\frac{w_i}{w_j} - 2.$$

Consistency Index

Basing on this method, Saaty introduced the Consistency Index:

$$CI(\mathbf{A}) = \frac{\lambda - n}{n - 1}.$$

REMARK

$CI(\mathbf{A}) = 0 \Leftrightarrow \mathbf{A}$ is consistent.

THEOREM (Aguarón and Moreno-Jiménez, 2003)

$$CI(\mathbf{A}) = \frac{1}{n(n-1)} \sum_{i<j} e_{ij},$$

where

$$e_{ij} = a_{ij}\frac{w_j}{w_i} + a_{ji}\frac{w_i}{w_j} - 2.$$

The algorithm of the input data improvement

Let $\mathbf{A} = [a_{ij}]$ be a pairwise comparison matrix. Fix a positive threshold ε.
The sketch of the algorithm of the input matrix improvement:

Repeat steps 1-2 until LCI($\mathbf{A}$) $< \varepsilon$.
Step 1: Find p, q such that $p < q$ and $e_{pq} = \max\limits_{i<j} e_{ij}$.
Step 2: Replace a_{pq} with $\frac{w_p}{w_q}$, and a_{qp} with $\frac{1}{a_{pq}}$.

The algorithm of the input data improvement

Let $\mathbf{A} = [a_{ij}]$ be a pairwise comparison matrix. Fix a positive threshold ε.
The sketch of the algorithm of the input matrix improvement:

Repeat steps 1-2 until LCI($\mathbf{A}$) $< \varepsilon$.
Step 1: Find p, q such that $p < q$ and $e_{pq} = \max_{i<j} e_{ij}$.
Step 2: Replace a_{pq} with $\frac{w_p}{w_q}$, and a_{qp} with $\frac{1}{a_{pq}}$.

EXAMPLE 1

INPUT:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & 1 \\ 0.25 & 1 & 1 \end{bmatrix}$$

and a threshold $\varepsilon = 0.01$.

Example 1

STEP 1:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 1 & 1 \\ 0.25 & 1 & 1 \end{bmatrix} \qquad \lambda_{MAX} = 3.217$$

$$[\tfrac{w_i}{w_j}] = \begin{bmatrix} 1 & 1.5874 & 2.5198 \\ 0.6300 & 1 & 1.5874 \\ 0.3969 & 0.6300 & 1 \end{bmatrix} \qquad w = \begin{bmatrix} 2.51984 \\ 1.5874 \\ 1 \end{bmatrix}$$

$$[e_{ij}] = \begin{bmatrix} 0 & 0.217361 & \boxed{0.217362} \\ * & 0 & 0.217361 \\ * & * & 0 \end{bmatrix} \qquad CI = 0.1085$$

STEP 2:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2.5198 \\ 1 & 1 & 1 \\ 0.3969 & 1 & 1 \end{bmatrix} \qquad \lambda_{MAX} = 3.09566$$

$$[\frac{w_i}{w_j}] = \begin{bmatrix} 1 & 1.3616 & 1.8518 \\ 0.7344 & 1 & 1.3600 \\ 0.5400 & 0.7353 & 1 \end{bmatrix} \qquad w = \begin{bmatrix} 1.85175 \\ 1.36 \\ 1 \end{bmatrix}$$

$$[e_{ij}] = \begin{bmatrix} 0 & \boxed{0.096021} & 0.095657 \\ * & 0 & 0.95294 \\ * & * & 0 \end{bmatrix} \qquad CI = 0.04783$$

Example 1

STEP 3:

$$\mathbf{A} = \begin{bmatrix} 1 & 1.3616 & 2.5198 \\ 0.7344 & 1 & 1 \\ 0.3969 & 1 & 1 \end{bmatrix} \qquad \lambda_{MAX} = 3.04225$$

$$\left[\frac{w_i}{w_j}\right] = \begin{bmatrix} 1 & 1.6717 & 2.0524 \\ 0.5982 & 1 & 1.2278 \\ 0.4872 & 0.8145 & 1 \end{bmatrix} \qquad w = \begin{bmatrix} 2.0524 \\ 1.22775 \\ 1 \end{bmatrix}$$

$$[e_{ij}] = \begin{bmatrix} 0 & 0.042246 & \boxed{0.0042250} \\ * & 0 & 0.042248 \\ * & * & 0 \end{bmatrix} \qquad CI = 0.021125$$

Example 1

STEP 4:

$$\mathbf{A} = \begin{bmatrix} 1 & 1.3616 & 2.0524 \\ 0.7344 & 1 & 1 \\ 0.4872 & 1 & 1 \end{bmatrix} \qquad \lambda_{MAX} = 3.01874$$

$$[\frac{w_i}{w_j}] = \begin{bmatrix} 1 & 1.5612 & 1.7900 \\ 0.6405 & 1 & 1.1466 \\ 0.5587 & 0.8722 & 1 \end{bmatrix} \qquad w = \begin{bmatrix} 1.79001 \\ 1.14658 \\ 1 \end{bmatrix}$$

$$[e_{ij}] = \begin{bmatrix} 0 & \boxed{0.0187405} & 0.00187395 \\ * & 0 & 0.0187389 \\ * & * & 0 \end{bmatrix} \qquad CI = 0.00937 < \varepsilon$$

EXAMPLE 2

INPUT:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 5 & 7 \\ 0.5 & 1 & 3 & 4 \\ 0.2 & 0.33 & 1 & 1 \\ 0.143 & 0.25 & 1 & 1 \end{bmatrix}$$

and a threshold $\varepsilon = 0.001$.

Example 2

STEP 1:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 5 & 7 \\ 0.5 & 1 & 3 & 4 \\ 0.2 & 0.333 & 1 & 1 \\ 0.143 & 0.25 & 1 & 1 \end{bmatrix} \qquad N = 1$$

$$[\frac{w_i}{w_j}] = \begin{bmatrix} 1 & 1.850 & 5.681 & 6.641 \\ 0.541 & 1 & 3.072 & 3.591 \\ 0.176 & 0.326 & 1 & 1.169 \\ 0.151 & 0.278 & 0.855 & 1 \end{bmatrix} \qquad w = \begin{bmatrix} 6.64129 \\ 3.59073 \\ 1.16903 \\ 1 \end{bmatrix}$$

$$[e_{ij}] = \begin{bmatrix} 0 & 0.0061 & 0.0163 & 0.0028 \\ * & 0 & 0.0006 & 0.0012 \\ * & * & 0 & \boxed{0.0244} \\ * & * & * & 0 \end{bmatrix} \qquad CI = 0.0052$$

Example 2

STEP 2:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 5 & 7 \\ 0.5 & 1 & 3 & 4 \\ 0.2 & 0.333 & 1 & 1.169 \\ 0.143 & 0.25 & 0.855 & 1 \end{bmatrix} \qquad N = 2$$

$$[\tfrac{w_i}{w_j}] = \begin{bmatrix} 1 & 1.849 & 5.474 & 6.922 \\ 0.541 & 1 & 2.961 & 3.744 \\ 0.183 & 0.338 & 1 & 1.264 \\ 0.144 & 0.267 & 0.791 & 1 \end{bmatrix} \qquad w = \begin{bmatrix} 6.92182 \\ 3.74352 \\ 1.26442 \\ 1 \end{bmatrix}$$

$$[e_{ij}] = \begin{bmatrix} 0 & 0.0062 & \boxed{0.0082} & 0.0001 \\ * & 0 & 0.0002 & 0.0044 \\ * & * & 0 & 0.0062 \\ * & * & * & 0 \end{bmatrix} \qquad CI = 0.0021$$

STEP 3:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 5.474 & 7 \\ 0.5 & 1 & 3 & 4 \\ 0.183 & 0.333 & 1 & 1.169 \\ 0.143 & 0.25 & 0.855 & 1 \end{bmatrix} \qquad N = 3$$

$$[\tfrac{w_i}{w_j}] = \begin{bmatrix} 1 & 1.890 & 5.728 & 7.075 \\ 0.529 & 1 & 3.031 & 3.744 \\ 0.175 & 0.330 & 1 & 1.235 \\ 0.141 & 0.267 & 0.810 & 1 \end{bmatrix} \qquad w = \begin{bmatrix} 7.0753 \\ 3.7437 \\ 1.235 \\ 1 \end{bmatrix}$$

$$[e_{ij}] = \begin{bmatrix} 0 & 0.0032 & 0.0021 & 0.0001 \\ * & 0 & 0.0001 & \boxed{0.0044} \\ * & * & 0 & 0.0030 \\ * & * & * & 0 \end{bmatrix} \qquad CI = 0.0011$$

Example 2

STEP 4:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 5.474 & 7 \\ 0.5 & 1 & 3 & 3.744 \\ 0.183 & 0.333 & 1 & 1.169 \\ 0.143 & 0.267 & 0.855 & 1 \end{bmatrix} \qquad N = 4$$

$$[\tfrac{w_i}{w_j}] = \begin{bmatrix} 1 & 1.922 & 5.728 & 6.961 \\ 0.520 & 1 & 2.980 & 3.621 \\ 0.175 & 0.336 & 1 & 1.215 \\ 0.144 & 0.276 & 0.823 & 1 \end{bmatrix} \qquad w = \begin{bmatrix} 6.9612 \\ 3.6213 \\ 1.215 \\ 1 \end{bmatrix}$$

$$[e_{ij}] = \begin{bmatrix} 0 & 0.0016 & \boxed{0.0021} & 0.0000 \\ * & 0 & 0.0000 & 0.0015 \\ * & * & 0 & 0.0015 \\ * & * & * & 0 \end{bmatrix} \qquad CI = 0.0006 < \varepsilon$$

Advantages

Advantages of the algorithm:

- simplicity;
- speed;
- preservation of most elements of the matrix;
- possibility of application in other methods (Least Squares Method, Logarithmic Least Squares Method).

Advantages

Advantages of the algorithm:

- simplicity;
- speed;
- preservation of most elements of the matrix;
- possibility of application in other methods (Least Squares Method, Logarithmic Least Squares Method).

Advantages

Advantages of the algorithm:

- simplicity;
- speed;
- preservation of most elements of the matrix;
- possibility of application in other methods (Least Squares Method, Logarithmic Least Squares Method).

Advantages

Advantages of the algorithm:

- simplicity;
- speed;
- preservation of most elements of the matrix;
- possibility of application in other methods (Least Squares Method, Logarithmic Least Squares Method).

Future study

What is left to do?

- a formal proof of the algorithm correctness;
- more tests;
- application in other methods (Least Squares Method, Logarithmic Least Squares Method).

Future study

What is left to do?

- a formal proof of the algorithm correctness;
- more tests;
- application in other methods (Least Squares Method, Logarithmic Least Squares Method).

Future study

What is left to do?

- a formal proof of the algorithm correctness;
- more tests;
- application in other methods (Least Squares Method, Logarithmic Least Squares Method).

THANK YOU FOR YOUR ATTENTION!