ISAHP 2005, Honolulu, Hawaii, July 8-10, 2005

Flow and Potential in Logarithmic Least Squares Estimation of AHP

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Keywords: flow, potential, pairwise comparison, logarithmic least squares estimation

Summary: *In the general theory of flow and potential, flow is induced by potential difference, and it is shown that pairwise comparison flow is also induced by priority weight potential difference in the logarithmic least squares estimation(LLSE) of node priority weight of a pairwise comparison design graph. While in the electrical circuit network, Kirchhoff's Current Law(KCL) holds on a cutset basis, Kirchhoff's Voltage Law(KVL) holds on a tieset basis, and Ohm's Law holds on a link basis, but in the LLSE of pairwise comparison design graph, the conservation law of pairwise comparison flow(KCL-like plus Ohm-like law) holds on a cutset basis. Using this law(or set of equations) systematically, topological formulae for the expression of priority weight potential are given for some design graphs.*

1. Introduction

In the general theory of flow and potential, flow is induced by potential difference, and this paper shows that pairwise comparison flow is also induced by priority weight potential difference in the logarithmic least squares estimation(LLSE) of node priority weight of a pairwise comparison design graph.

Rest of the paper consists as follows.

Chapter2: LLSE of node priority weight
Chapter3: Optimality condition of LLSE
Chapter4: Flow and potential in LLSE
Chapter5: Examples
Chapter6: Topological formula
Chapter7: Conclusion

2. LLSE of node priority weight

2.1 Pairwise comparison design graph

We will introduce pairwise comparison design graph G(V, E) to express what pairwise comparison is made among the items. The graph G (V, E) is a directed-edge graph, V is the vertex set, and E is the edge set. An item to be evaluated corresponds to a vertex, and if there is an edge between two vertexes, then it means that a pairwise comparison measurement took place between the two items. Multiple edges are allowed between two vertexes.

2.2 Ratio model and its logarithmic linear model

We assume two ratio models (1) and (2) for the pairwise comparison measurement between item i and item j.

$$\left(a_{ij}\right)^{R_{ij}} = \frac{x_i}{x_j} e_{ij} \qquad (i, j) \in E \qquad (1)$$

$$a_{ij} = \left(\frac{x_i}{x_j}\right)^{G_{ij}} e_{ij} \qquad (i, j) \in E \qquad (2)$$

Here, a directed edge (i, j) originates at vertex i and terminates at vertex j. If we take the logarithm of both sides in (1) and (2), then the logarithmic linear models (2) and (3) are obtained.

$$R_{ij}\alpha_{ij} = u_i - u_j + \varepsilon_{ij} \qquad (i, j) \in E \qquad (3)$$

$$\alpha_{ij} = \left(u_i - u_j\right)G_{ij} + \varepsilon_{ij} \qquad (i, j) \in E \qquad (4)$$

Here, (i, j) is an edge connecting item i and item j, a_{ij} is pairwise comparison measurement value when item i is compared how many times more important than item j (a_{ij} is simply called measurement (i, j)), x_i is the weight of item i, e_{ij} is the error multiplier accompanied with measurement (i, j), R_{ij} is the tendency of overestimation or underestimation accompanied with measurement (i, j), G_{ij} is also the tendency of overestimation or underestimation accompanied with measurement (where $R_{ij} \approx G_{ij}^{-1}$), $\alpha_{ij} = \log a_{ij}$, $u_i = \log x_i$, $\varepsilon_{ij} = \log e_{ij}$, and so on. Hereafter a_{ij} is simply called measurement (i, j), and if multiple measurements are allowed on the same edge (i, j), the third index would be needed to distinguish the measurements done on the same edge (i, j), such as by (i, j;1), (i, j;2), and (i, j;3). But for the simplicity of notation the third index will be neglected.

Notice that (1) and (2) are not linear in x, but (3) and (4) are linear in α and u, given R and G.

3. Optimality condition of LLSE

3.1 Formulation of LLSE

The problem of estimating the priority weight vector of items $u = \{u_i\}$ so as to minimize the sum of logarithmic errors is mathematically formulated as

$$\text{minimize} \qquad Z(u) = \sum \left(\varepsilon_{ij}\right)^2 \qquad (5)$$

$$\text{subject to} \qquad \sum u_i = K \qquad (6)$$

, where the summation $\sum$ in (5) is taken over $(i, j) \in E$, and K is an appropriate constant.

3.2 Optimality condition

Since the LLSE problem ((5) and (6)) is an equality-constrained minimization problem, its necessary condition for the optimality is obtained by differentiating the Lagrange function L (u, λ) with respect to u and λ and putting each of them at zero.

$$L(u, \lambda) = Z(u) + \lambda\left(\sum u_i - K\right) \qquad (7)$$

$$\frac{\partial L}{\partial u_i} = 0 \qquad i \in V \qquad (8)$$

$$\frac{\partial \mathrm{L}}{\partial \lambda} = 0 \qquad (9)$$

The set of equations (8) and (9) is the optimality condition for the LLSE. Then, the following theorems hold.

〔Theorem 1〕

The Lagrange multiplier λ for the constraint (6) is 0 both in Model (1) and model (2). □

〔Proof〕 First, we consider the case of Model (1).

Since it holds that

$$\frac{\partial \mathrm{L}}{\partial u_i} = -2\sum_{k\in T(i)}\left(\mathrm{R}_{ik}\alpha_k^{ik} - (u_i - u_k)\right) + 2\sum_{k\in o(i)}\left(\mathrm{R}_{ki}\alpha_{ki} - (u_k - u_i)\right) + \lambda \qquad (i \in \mathrm{V}) \quad (10)$$

and $\frac{\partial \mathrm{L}}{\partial u_i} = 0$ ($i \in \mathrm{V}$), summing up (10) for each $i \in \mathrm{V}$ and putting it at zero,

we get $N\,\lambda = 0$. Here, $N = |\mathrm{V}|$, $T(i) = \{k/(i,k) \in E\}$, $O(i) = \{k/(k,i) \in E\}$.

Notice that the term corresponding to some specific edge (l,m) appears twice, once in the first term of any equation and the second time in the second term of another equation with different signs, thus canceling each other. Next, we consider the case of Model (2). Instead of (10), we have (11).

$$\frac{\partial \mathrm{L}}{\partial u_i} = -2\sum_{k\in T(i)}\mathrm{G}_{ik}\left(\alpha_{ik} - \mathrm{G}_{ik}(u_i - u_k)\right) + 2\sum_{k\in 0(i)}\mathrm{G}_{ki}\left(\alpha_{ki} - \mathrm{G}_{ki}(u_k - u_i)\right) + \lambda \qquad (i \in \mathrm{V}) \qquad (11)$$

The same argument goes and finally it holds that $N\,\lambda = 0$. (Q.E.D.)

〔Theorem 2〕

In Model (1), the following inflow = outflow conservation-like law holds.

$$\sum_{k\in T(i)}\mathrm{R}_{ik}\alpha_{ik} - \sum_{k\in 0(i)}\mathrm{R}_{ki}\alpha_{ki} = \sum_{k\in T(i)}(u_i - u_k) - \sum_{k\in 0(i)}(u_k - u_i) \qquad (i \in \mathrm{V}) \qquad (12)$$

Assuming the reciprocity, ($a_{ij} \cdot a_{ji} = 1$), it can be expressed in an undirected-edge manner by (13).

$$\sum_{k\in A(i)}\mathrm{R}_{ik}\alpha_{ik} = \sum_{k\in A(i)}(u_i - u_k) \qquad (i \in \mathrm{V}) \qquad (13)$$

Here, $A(i) = \{k \mid (i,k) \in E \text{ or } (k,i) \in E\}$. □

〔Proof〕 Since $\lambda = 0$ in (10), (12) is directly induced. (Q.E.D.)

Note that a simple formula like (13) does not hold in Model (2). So Model (1) is preferred to Model (2) also from the viewpoint of theoretical beauty.

4. Flow and potential in LLSE

In resistor electrical networks, the Ohm's Law holds on an edge, such as RI＝V, Kirchhoff's Current Law (KCL) holds on a cutset, and Kirchhoff's Voltage Law (KVL) holds on a tieset. In LLSE of pairwise comparison graph, what kind of laws will hold and will not hold?

4.1 KVL

In an electrical network, KVL says that edge voltage integrated along any loop（tieset）is zero.

Since variable u_i is assigned to each node and its difference ($u_i - u_j$) appears in its error model (3), consider it as its potential. Then, ($u_i - u_j$) is the potential difference between nodes i and j, and can be called "voltage". Since the potential is assigned to each node and their difference between nodes can be considered voltage, a KVL-like law naturally holds in our LLSE of pairwise comparison graph.

4.2 Ohm's Law + KCL

Equation (13) can be interpreted as a combination of Ohm's Law and KCL. Since α_{ik} is the logarithm of pairwise measurement (i,k), it can be interpreted as the evaluation flow oriented from node i to node k. The left hand side of (13) or (12) is summation of all the evaluation flows coming to and going from node i, and the right hand side of (13) or (12) is the summation of potential difference. If Eq. (13) holds on an edge basis, it is Ohm's Law, but Eq.(13) does hold on a node basis. Therefore, Theorem 2 can be extended to Theorem 3.

〔Theorem 3〕

In Model (1), inflow＝outflow conservation-like law holds at any cutest C.

$$\sum_{(i,j)\in \mathrm{C}} \mathrm{R}_{ij}\alpha_{ij} = \sum_{(i,j)\in \mathrm{C}} (u_i - u_j) \tag{14}$$

5. Examples

We have shown the concept of evaluation flow and evaluation potential in Chapter 4 and we will explain them through examples in this chapter.

〔Example 1：Model (1) with $\mathrm{R}_{ij} = 1$〕

Consider a design graph of Fig.1, where there are 4 items and 5 pairwise comparison measurements take place. They are measurements (1,2), (2,3), (1,3), (1,4) and (3,4);

$a_{12} = 1,\ a_{23} = 2,\ a_{13} = 2,\ a_{14} = 5,\ a_{34} = 6,\ and\ \mathrm{R}_{ij} = 1$ for all $(i,j)_s \in E$.

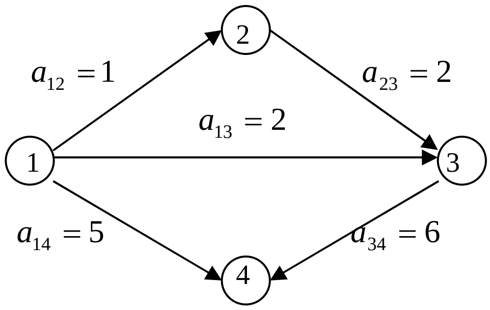

Fig.1 Four-node five-edge design graph for example 1

LLSE solutions are:

$$x_1 = 1.7783,\ x_2 = 1.9839,\ x_3 = 1.1067,\ x_4 = 0.2561$$
$$u_1 = 0.25,\ u_2 = 0.2975,\ u_3 = 0.044,\ u_4 = -0.5915.$$

Here, u_1, u_2, u_3, and u_4 are interpreted as evaluation potentials and they are shown on the undirected-edge version of the design graph (Fig. 2), together with evaluations flows $\alpha_{12} = 0.0$, $\alpha_{13} = 0.301,\ \alpha_{23} = 0.301,\ \alpha_{14} = 0.69897,\ and\ \alpha_{34} = 0.778$.

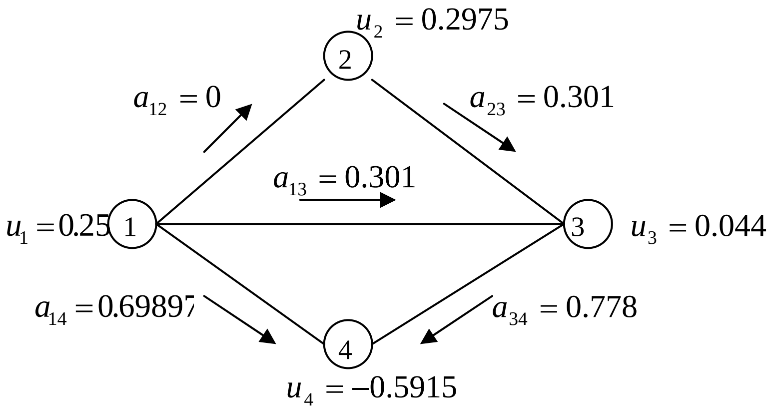

Fig.2 Flows and potentials for Example 1

First, consider a cutest $c_1 = \{(1,2),(1,3),(1,4)\}$, the set of edges connecting vertex 1 and the other vertexes. The sum of evaluation flows through c_1 is $\alpha_{12} + \alpha_{13} + \alpha_{14}$, which is 1.0, and the sum of evaluation potential differences at c_1 is $(u_1 - u_2) + (u_1 - u_3) + (u_1 - u_4)$, which is also 1.0. Therefore, Theorem 2 and Theorem 3 are confirmed to hold for the cutest c_1 in Example 1.

Next, consider a cutest $c_2 = \{(1,2),(1,3),(4,3)\}$, the set of edges connecting vertex set {1, 4} and vertex set {2, 3}. The sum of evaluation flows through c_2 is $\alpha_{12} + \alpha_{13} - \alpha_{34}$, which is –0.477, and the sum of evaluation potential differences at c_2 is $(u_1 - u_2) + (u_1 - u_3) + (u_4 - u_3)$, which is also –0.477. Therefore, Theorem 3 is confirmed to hold also for the cutest c_2 in Example 1.

〔Example 2: Model (1) with different $\mathrm{R}_{ij's}$〕

Consider a design graph shown in Fig.3, where measurement values are the same as in Example 1, but $\mathrm{R}_{ij's}$ are not all equal to 1;

$$\mathrm{R}_{12} = 1,\ \mathrm{R}_{23} = 1,\ \mathrm{R}_{13} = 2,\ \mathrm{R}_{14} = 1,\ \mathrm{R}_{34} = 2.$$

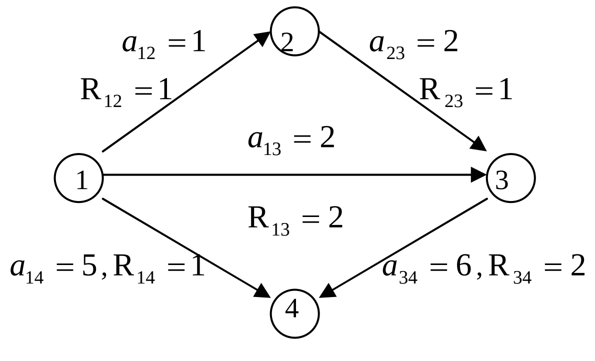

Fig.3 Four-node five-edge design graph for Example 2

LLSE solutions are:

$$x_1 = 2.1147,\ x_2 = 2.482,\ x_3 = 1.4565,\ x_4 = 0.1308$$

$$\mathrm{u}_1 = 0.3252,\ u_2 = 0.3948,\ u_3 = 0.1633,\ u_4 = -0.8833.$$

Evaluation potentials u_1, u_2, u_3, and u_4, and evaluation flows $\alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{14}$ and α_{34}, are shown on the undirected-edge version of the design graph (Fig.4), together with resistance values $R_{12}, R_{13}, R_{23}, R_{14}$ and R_{34}.

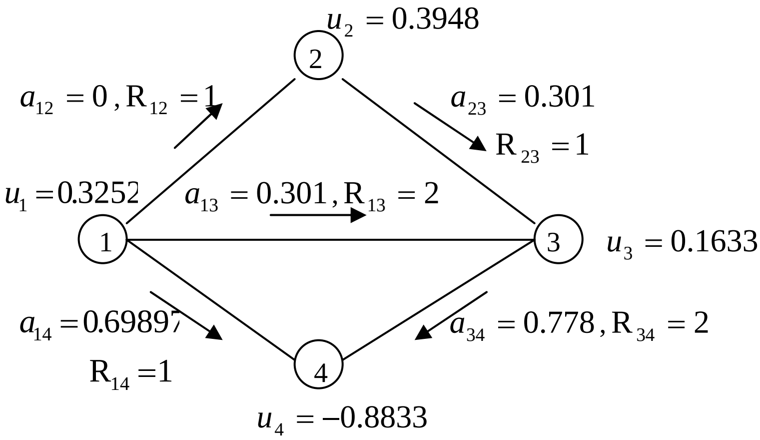

Fig.4 Flows and potentials for Example 2

First, consider a cutest $c_1 = \{(1, 2), (1, 3), (1, 4)\}$, the set of edges connecting vertex 1 and the other vertexes. The sum of evaluation flows through c_1, multiplied by each resistance value, is $R_{12}\alpha_{12} + R_{13}\alpha_{13} + R_{14}\alpha_{14}$, which is 1.3, and the sum of evaluation potential differences at c_1 is $(u_1 - u_2) + (u_1 - u_3) + (u_1 - u_4)$, which is also 1.3. Therefore, Theorem 2 and Theorem 3 are confirmed to hold for the cutest 1 in Example 2.

Next, consider a cutest $c_2 = \{(1, 2), (1, 3), (4, 3)\}$, the set of edges connecting vertex set {1,4} and vertex set {2,3}. The sum of evaluation flows through c_2, multiplied by each resistance value, is $R_{12}\alpha_{12} + R_{13}\alpha_{13} - R_{34}\alpha_{34}$, which is –0.954, and the sum of evaluation potential differences at c_2 is $(u_1 - u_2) + (u_1 - u_3) + (u_4 - u_3)$, which is also –0.954. Therefore, Theorem 3 is confirmed to hold also for the cutest c_2 in Example 2.

6. Topological formula

In this chapter we will consider some classes of design graphs with no multiple edge and of Model (1) with all $R_{ij's}$ being equal to 1, and derive topological formulae for the expression of prioritoy weight potentials.

6.1 Complete design graph

For a complete graph with N vertexes, Theorem 3 holds at each node cutest.

$$\sum_{j=1}^{N} \alpha_{ij} = \sum_{j=1}^{N} (u_i - u_j) \qquad i = 1, \ldots, N \qquad (15)$$

Noting $\sum_{j=1}^{N} u_j = 0$, (16) or (17) is obtained, which is the well-known geometric mean formula for LLSE solution.

$$u_i = \frac{1}{N}\sum_{j=1}^{N}\alpha_{ij} \qquad (16)$$

$$x_i = \left(\prod_{j=1}^{N} a_{ij}\right)^{\frac{1}{N}} \qquad (17)$$

Here, notice that $\alpha_{ij} + \alpha_{ji} = 0$ or $a_{ij} \cdot a_{ji} = 1$ because of the reciprocity assumption, and that either measurement (i, j) or measurement (j, i) takes place because no multiple edge is allowed in the design graph.

6.2 Tree design graph

For a tree graph, Theorem 3 holds at each edge cutest.

$$\alpha_{ij} = u_i - u_j \qquad (i, j) \in E \qquad (18)$$

Choose any vertex, say vertex k, as a reference point $(u_k = 0)$. Since the topology of design graph is tree, there exists only one path, or one chain of edges from vertex *i* to vertex *j*, let the chain be $\{(i, j_1), (j_1, j_2), ..., (j_m, k)\}$, then (19) or (20) is obtained by applying Eq.(18) successively along the chain $\{(i, j_1), (j_1, j_2), ..., (j_m, k)\}$.

$$u_i = \alpha_{ij_1} + \alpha_{j_1 j_2} + ... + \alpha_{j_m k} \qquad (19)$$

$$x_i = a_{ij_1} \cdot a_{j_1 j_2} \cdot ... \cdot a_{j_m k} \qquad (20)$$

6.3 1-cyclic design graph

Consider a design graph with N vertexes where edges are arranged as (1, 2), (2, 3), (3, 4),…,(N −1, N) and (N , 1). This class of graphs is called “cycle” or “loop”. Any simple cutest contains two edges, say (i, j) and (k, l). Then, Theorem 3 holds at the cutest.

$$\alpha_{ij} + \alpha_{kl} = (u_i - u_j) + (u_k - u_l) \qquad ((i, j), (k, l) \in E) \qquad (21)$$

Rearranging these equations and setting vertex N as a reference point $(u_N = 0)$, following formula is obtained.

$$u_i = \alpha_{i,i+1} + \alpha_{i+1,i+2} + ... + \alpha_{N-1,N} + \frac{N-i}{N}\Delta \qquad (22)$$

$$\Delta = \alpha_{N,N-1} + \alpha_{N-1,N-2} + ... + \alpha_{32} + \alpha_{21} + \alpha_{1N} \qquad (23)$$

$$x_i = a_{i,i+1} \cdot a_{i+1,i+2} \cdot ... \cdot a_{N-1,N} \cdot \delta^{\frac{N-i}{N}} \qquad (24)$$

$$\delta = a_{N,N-1} \cdot a_{N-1,N-2} \cdot ... \cdot a_{32} \cdot a_{21} \cdot a_{1N} \qquad (25)$$

6.4 Complete graph minus one edge

Consider a design graph where one edge is deleted from a complete graph. Let the graph has N vertexes and the deleted edge be $(1, N)$. Then, Theorem 2 or Theorem 3 holds at vertex i$(\neq 1, N)$.

$$\sum_{j=1}^{N}\alpha_{ij} = \sum_{j=1}^{N}\left(u_i - u_j\right) \qquad \left(i \neq 1, N\right) \qquad (26)$$

Therefore,

$$u_i = \frac{1}{N}\sum_{j=1}^{N}\alpha_{ij} \qquad \left(i \neq 1, N\right) \qquad (27)$$

or $$x_i = \left(\prod_{j=1}^{N} a_{ij}\right)^{\frac{1}{N}} \qquad \left(i \neq 1, N\right) \qquad (28)$$

So the geometric mean formula works for the priority weight potential for an item with complete matching. For the vertexes 1 and N, following equations hold.

$$\left(N-2\right)\left(u_1 - u_N\right) = \sum_{k=2}^{N-1}\left(\alpha_{1k} + \alpha_{kN}\right) \qquad (29)$$

$$\left(N-2\right)\left(u_N - u_1\right) = \sum_{k=2}^{N-1}\left(\alpha_{Nk} + \alpha_{k1}\right) \qquad (30)$$

$$Nu_1 = \sum_{k=1}^{N-1}\alpha_{1k} + \left(u_1 - u_N\right) \qquad (31)$$

$$Nu_N = \sum_{k=2}^{N}\alpha_{Nk} + \left(u_N - u_1\right) \qquad (32)$$

Therefore, we have (31), (32), (33) and (34).

$$u_1 = \frac{1}{N}\left\{\sum_{k=1}^{N-1}\alpha_{1k} + \frac{1}{N-2}\sum_{k=2}^{N-1}\left(\alpha_{1k} + \alpha_{kN}\right)\right\} \qquad (31)$$

$$u_N = \frac{1}{N}\left\{\sum_{k=1}^{N-1}\alpha_{Nk} + \frac{1}{N-2}\sum_{k=2}^{N-1}\left(\alpha_{Nk} + \alpha_{k1}\right)\right\} \qquad (32)$$

$$x_1 = \left\{\left(\prod_{k=1}^{N-1} a_{1k}\right) \times \left(\prod_{k=2}^{N-1} a_{1k} \cdot a_{kN}\right)^{\frac{1}{N-2}}\right\}^{\frac{1}{N}} \qquad (33)$$

$$x_N = \left\{\left(\prod_{k=1}^{N-1} a_{Nk}\right) \times \left(\prod_{k=2}^{N-1} a_{Nk} \cdot a_{k1}\right)^{\frac{1}{N-2}}\right\}^{\frac{1}{N}} \qquad (34)$$

〔Example 3〕 Applying the formulae (28), (33) and (34) to the four-node five-edge design graph of Example 1 (Fig.1), the followings are obtained.

$$x_1 = \left(a_{11}\ a_{12}\ a_{13}\ a_{14}\right)^{\frac{1}{4}} \qquad (35)$$

$$x_2 = \left(a_{21}\ a_{22}\ a_{23}\ a_{24}\right)^{\frac{1}{4}} \qquad (36)$$

$$A_{24} = \left(a_{21}a_{14}\right)^{\frac{1}{2}}\left(a_{23}a_{34}\right)^{\frac{1}{2}} \qquad (37)$$

$$x_3 = \left(a_{31}\ a_{32}\ a_{33}\ a_{34}\right)^{\frac{1}{4}} \qquad (38)$$

$$x_4 = (a_{41}\ a_{42}\ a_{43}\ a_{44})^{\frac{1}{4}} \quad (39)$$

$$A_{42} = (a_{41}a_{12})^{\frac{1}{2}}(a_{43}a_{32})^{\frac{1}{2}} \quad (40)$$

Inserting $a_{12} = 1,\ a_{23} = 2,\ a_{13} = 2,\ a_{14} = 5,\ a_{34} = 6$ and assuming the reciprocity, $x_1 = 1.7783,\ x_2 = 1.9839,\ x_3 = 1.1067,\ x_4 = 0.2561$ and obtained, which coincide with those in Example 1.

7. Conclusion

We have introduced the concept of evaluation flow and evaluation potential. Pairwise comparison corresponds to flow and priority weight corresponds to potential. It is shown that evaluation flow is induced by evaluation potential difference at a cutest, which can be interpreted as a cutset-version of KCL ＋Ohm's law in the electrical resistance-circuit network. Applying the obtained laws and equations, we have also presented some topological formulae for priority weight, which will be useful in investigating the meaning of LLSE solution.

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