ISAHP 2005, Honolulu, Hawaii, July 8-10, 2005

A method to speedily pairwise compare in AHP and ANP

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Keywords: Speedily pairwise comparisons, 2-cluster ANP, Binary AHP

Summary: *AHP (Analytic Hierarchy Process) and ANP (Analytic Network Process) are useful tool for decision makers. However, the amount of pairwise comparison becomes large with increasing the number of alternatives and criteria. Therefore, it takes much time and the loads of the decision maker increase. In AHP and ANP, it is important to pairwise compare carefully but to need speedily. This paper proposes a method to deal speedily with pairwise comparisons, and apply our method to AHP and 2-cluster ANP. In our method, we introduce three-level evaluation, scale values range from 0 to 2. At first, decision maker sets the standard in each criterion, and next, we evaluate alternatives and construct the matrix, called evaluation matrix. In AHP, based on evaluation matrix, we construct comparison matrix automatically and have each weight of alternative. In 2-cluster ANP, equivalent to AHP, we construct super matrix automatically and have each weight. The usefulness of our method was confirmed through some examples.*

1. Introduction

AHP (Analytic Hierarchy Process) (Saaty, 1980) and ANP (Analytic Network Process) (Saaty, 2001) are useful tool for decision makers. However, the amount of pairwise comparison becomes large with increasing the number of alternatives and criteria. Therefore, it takes much time and the loads of the decision maker increase.

In AHP and ANP, it is important to pairwise compare carefully but to need speedily. To implement AHP in decision-making part of several software, it is indispensable to construct comparison matrix and super matrix speedily and automatically. This paper proposes a method to deal speedily with pairwise comparisons, and apply our method to AHP and 2-cluster ANP.

In our method, we introduce three-level evaluation, scale values range from 0 to 2. At first, decision maker sets the standard in each criterion, and next, we evaluate alternatives and construct the matrix, called evaluation matrix. For each criterion, if alternative satisfy the standard then let evaluation value be 2, else be 0, but unknown be 1.

In AHP, based on evaluation matrix, we construct comparison matrix automatically and have each weight of alternative. In 2-cluster ANP, equivalent to AHP, we construct super matrix automatically and have each weight. To determine these weights, in our method, we introduce binary AHP (Takahashi, 1990).

In this paper, the procedure of speedily pairwise comparison is described in section 2. In section 3, illustrates an example by proposed methods. Finally conclude in this investigation in section 4.

2. Procedure

In this section, the procedure of speedily pairwise comparison is proposed. Assume decision maker prioritize n alternatives based on m criteria.

At first, decision maker selects m criteria, and sets the standard in each criterion carefully and next evaluate alternatives. In our method, we introduce three-level evaluation, scale values range from 0 to 2. The details are shown in Table 1.

Table 1 Three-level evaluation

Standard of Criterion	Evaluation Value
satisfy	2
not satisfy	0
unknown	1

Next, to prepare for our method, we construct the matrix D, called evaluation matrix. We denote the element of the matrix D by d_{ij} (i = 1 to m and j=1 to n). For example, if alternative a_j satisfy the standard of criterion c_i then let be d_{ij}=2 else be d_{ij}=0, but unknown be d_{ij}=1.

Based on matrix D, we describe our method to speedily construct comparison matrix and super matrix, respectively.

2.1 Procedure in AHP

In AHP, we denote the weights of alternatives by w and calculate by following equation.

$$w = WV \tag{1}$$

As well known the matrix W consists of the column vector Wc_i (i=1 to m), as follows:

$$W = \begin{bmatrix} Wc_1 & Wc_2 & \cdots & Wc_m \end{bmatrix}. \tag{2}$$

Wc_i is the weight of alternatives with respect to criterion c_i, from comparison matrix Ac_i.

To construct comparison matrix, based on D, we introduce binary comparison. In our method, we denote the element of comparison matrix Ac_i by a_{xy}(x,y = 1 to n) and $a_{xy}=\theta(\, d_{ix} > d_{iy})$, $a_{yx}=1/a_{xy}$, $a_{xx}=1$ and $\theta(>1)$ is a parameter.

To construct W, proposed procedure is summarized, as follows:

(W-0): i=0.
(W-1): Add 1 to i.
(W-2): Pairwise binary compare with d_{ij}, ith-row vector of D**,** for j=1 to n.
(W-3): Construct comparison Matrix Ac_i.
(W-4): Calculate Wc_i from Ac_i.
(W-5): Repeat (W-1) to (W-4) until i=m, and construct W.

Next, we construct V speedily. The vector V is the weights of criteria. To calculate V, in general, there are two kind of method. One is by ordinary pairwise comparison between criteria and the other is by calculate the weights automatically. In this paper, we introduce the latter method.

To prioritize criteria and have corresponding weights, in this paper, we use the frequency of evaluation value 2 (satisfied) on *D*. Furthermore there are two kind of concept, one is give a high priority to frequent of satisfy criterion and the other is give a high priority to infrequent criterion. There are various methods to decide the weights, for example by binary AHP and so on.

Finally, we can obtain *W* and *V*, above procedure, and have *w* from (1).

2.2 Procedure in 2-cluster-ANP

In 2-cluster ANP, equivalent to AHP, we construct super matrix *S* automatically and have each weight. Super matrix *S* of 2-cluster ANP consists of sub matrix *W*, *V*, and *O*, as follows:

$$S = \left[\begin{array}{c|c} O & V \\ \hline W & O \end{array}\right]. \tag{3}$$

From (3), we have the weights of alternatives *w* and the weights of criteria *v*.

In (3), *O* is the zero matrix. The sub matrix *W* is coinciding with (2). The sub matrix *V* consists of the column vector Va_j (j=1 to n), as follows:

$$V = \begin{bmatrix} Va_1 & Va_2 & \cdots & Va_n \end{bmatrix}. \tag{4}$$

Va_j is the weight of criterion for alternative a_j, from comparison matrix Aa_j.

If alternative a_j satisfy the standard of criterion c_x (d_{xj}=2) and not satisfy the standard of criterion c_y (d_{yj}=0), then we judge that c_x is more favor than c_y for a_j. So to construct Aa_j, we use jth-column vector of *D*.

To construct *V*, proposed procedure is summarized as follows:

(V-0): j=0.
(V-1): Add 1 to j.
(V-2): Pairwise binary compare with d_{ij}, jth-column vector of *D*, for i=1 to m.
(V-3): Construct comparison Matrix Aa_j.
(V-4): Calculate Va_j from Aa_j.
(V-5): Repeat (V-1) to (V-4) until j=n, and construct *V*.

From (3), we have the weights of alternatives *w*, and the weights of criteria *v*, simultaneously. To calculate *w* and *v* from super matrix, there are two kind of method. One is calculating the infinite power of *S* and the other is calculating the principle eigen vector corresponding to the eigen value 1. In this study, we calculate *w* and *v* by latter method.

3. Example

In this section, to explain proposed method, an example is illustrated. To order six alternatives (n=6; a_1 to a_6), decision maker selects five criteria (m=5; c_1 to c_5) and carefully sets the standard in each criterion.

For each standard, based on Table 1, we construct the matrix *D*, as follows:

$$D = \begin{bmatrix} 2 & 2 & 0 & 1 & 1 & 0 \\ 2 & 1 & 2 & 2 & 2 & 2 \\ 2 & 2 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 & 2 & 2 \end{bmatrix}. \quad (5)$$

For example, d_{13}=0, the element of the matrix D, means alternative a_3 not satisfy the standard of criterion c_1.

3.1 Example in AHP

Following proposed procedure (W-0) to (W-5), described in section 2.1, we construct comparison matrix Ac_1 to Ac_5 based on D and construct W. For example, with respect to criterion c_1, we use d_{1j}, 1*st*-row vector of D, as follows:

$$d_{1j} = \begin{bmatrix} 2 & 2 & 0 & 1 & 1 & 0 \end{bmatrix}. \quad (6)$$

Based on (6), we have comparison matrix Ac_1, as follows:

$$Ac_1 = \begin{bmatrix} 1 & 1 & \theta & \theta & \theta & \theta \\ 1 & 1 & \theta & \theta & \theta & \theta \\ 1/\theta & 1/\theta & 1 & 1/\theta & 1/\theta & 1 \\ 1/\theta & 1/\theta & \theta & 1 & 1 & \theta \\ 1/\theta & 1/\theta & \theta & 1 & 1 & \theta \\ 1/\theta & 1/\theta & 1 & 1/\theta & 1/\theta & 1 \end{bmatrix}. \quad (7)$$

Next, from (7), we calculate Wc_1, principle eigen vector of Ac_1, by power method where θ =2. The convergence limit in this method is 10^{-6}. The result of Wc_1 is shown as (8).

$$Wc_1 = \begin{bmatrix} 0.24669 \\ 0.24669 \\ 0.09790 \\ 0.15540 \\ 0.15540 \\ 0.09790 \end{bmatrix} \quad (8)$$

As similar to have Wc_1, we also have Wc_2 to Wc_5 and construct matrix W. The result of W is shown as below:

$$W = \begin{bmatrix} 0.24669 & 0.18181 & 0.22222 & 0.12500 & 0.18181 \\ 0.24669 & 0.09090 & 0.22222 & 0.12500 & 0.18181 \\ 0.09790 & 0.18181 & 0.11111 & 0.12500 & 0.18181 \\ 0.15540 & 0.18181 & 0.11111 & 0.25000 & 0.09090 \\ 0.15540 & 0.18181 & 0.11111 & 0.25000 & 0.18181 \\ 0.09790 & 0.18181 & 0.22222 & 0.12500 & 0.18181 \end{bmatrix}. \quad (9)$$

Next, we calculate V based on frequency of evaluation value. Frequency of evaluation value, in this example, is shown in Table 2.

Table 2 Frequency of Evaluation Value

criterion\ value	2	1	0
c_1	2	2	2
c_2	5	1	0
c_3	3	3	0
c_4	2	0	4
c_5	5	0	1

In this study, we calculate infrequent case and frequent case, respectively. First, we calculate $\boldsymbol{V}$ based on infrequent of satisfy the standard of each criterion. From Table 2, we have the infrequent order of criteria, $c_4 > c_1 > c_3 > c_5 > c_2$. Based on above order, we construct binary comparison matrix V_l, as follows:

$$V_l = \begin{bmatrix} 1 & \theta & \theta & 1/\theta & \theta \\ 1/\theta & 1 & 1/\theta & 1/\theta & 1/\theta \\ 1/\theta & \theta & 1 & 1/\theta & \theta \\ \theta & \theta & \theta & 1 & \theta \\ 1/\theta & \theta & 1/\theta & 1/\theta & 1 \end{bmatrix}. \tag{10}$$

From (10), where θ =2, we have v_l, as follows:

$$v_l = \begin{bmatrix} 0.244679 \\ 0.106503 \\ 0.185432 \\ 0.322856 \\ 0.140531 \end{bmatrix}. \tag{11}$$

From (1), (9) and (11), we can obtain w_l, as follows:

$$w_l = \begin{bmatrix} 0.187332 \\ 0.175436 \\ 0.133652 \\ 0.166759 \\ 0.182456 \\ 0.154364 \end{bmatrix}. \tag{12}$$

From (12), as a result in infrequent case, we have the order of alternatives, $a_1 > a_5 > a_2 > a_4 > a_6 > a_3$.

On the other hand, we have the order of criteria in frequent, $c_2 > c_5 > c_3 > c_1 > c_4$. As similar to infrequent case, we have v_h and w_h, as follows:

$$v_h = \begin{bmatrix} 0.140531 \\ 0.322856 \\ 0.185432 \\ 0.106503 \\ 0.244679 \end{bmatrix}, \quad (13)$$

$$w_h = \begin{bmatrix} 0.191202 \\ 0.158601 \\ 0.155360 \\ 0.148023 \\ 0.172730 \\ 0.174084 \end{bmatrix}. \quad (14)$$

From (14), as a result in frequent case, we have the order of alternatives, $a_1 > a_6 > a_5 > a_2 > a_3 > a_4$.

The result, the order of criterion and alternatives in infrequent case and frequent case are summarized in Table 3.

Table 3 The order of criterion and alternatives in AHP

	The order of criterion					The order of alternative					
Infrequent	c_4	c_1	c_3	c_5	c_2	a_1	a_5	a_2	a_4	a_6	a_3
Frequent	c_2	c_5	c_3	c_1	c_4	a_1	a_6	a_5	a_2	a_3	a_4

In Table 3, the order of alternative a_1 is high even if the order of criteria is different.

3.2 Example in 2-cluster-ANP

Following proposed procedure (V-0) to (V-5), described in section 2.2, we construct comparison matrix Aa_1 to Aa_6 based on D and construct V. For example Aa_1, we use d^T_{i1}, 1*st*-column vector of D, as follows:

$$d^T{}_{i1} = [2 \quad 2 \quad 2 \quad 0 \quad 2]. \quad (15)$$

Based on (15), we have comparison matrix Aa_1, as follows:

$$Aa_1 = \begin{bmatrix} 1 & 1 & 1 & \theta & 1 \\ 1 & 1 & 1 & \theta & 1 \\ 1 & 1 & 1 & \theta & 1 \\ 1/\theta & 1/\theta & 1/\theta & 1 & 1/\theta \\ 1 & 1 & 1 & \theta & 1 \end{bmatrix}. \quad (16)$$

From (16), we have Va_1 where θ=2, as follows:

$$Va_1 = \begin{bmatrix} 0.222222 \\ 0.222222 \\ 0.222222 \\ 0.111111 \\ 0.222222 \end{bmatrix}. \quad (17)$$

As similar to construct W, we have Va_2 to Va_6 and construct matrix V, as follows:

$$V = \begin{bmatrix} 0.222222 & 0.248182 & 0.122622 & 0.162803 & 0.125000 & 0.125000 \\ 0.222222 & 0.145773 & 0.282583 & 0.282583 & 0.250000 & 0.250000 \\ 0.222222 & 0.248182 & 0.189591 & 0.162803 & 0.125000 & 0.250000 \\ 0.111111 & 0.109682 & 0.122622 & 0.282583 & 0.250000 & 0.125000 \\ 0.222222 & 0.248182 & 0.282583 & 0.109228 & 0.250000 & 0.250000 \end{bmatrix}. \quad (18)$$

As a result, we have super matrix S of this example, as follows:

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.22222 & 0.24818 & 0.12262 & 0.16280 & 0.12500 & 0.12500 \\ 0 & 0 & 0 & 0 & 0 & 0.22222 & 0.14577 & 0.28258 & 0.28258 & 0.25000 & 0.25000 \\ 0 & 0 & 0 & 0 & 0 & 0.22222 & 0.24818 & 0.18959 & 0.16280 & 0.12500 & 0.25000 \\ 0 & 0 & 0 & 0 & 0 & 0.11111 & 0.10968 & 0.12262 & 0.28258 & 0.25000 & 0.12500 \\ 0 & 0 & 0 & 0 & 0 & 0.22222 & 0.24818 & 0.28258 & 0.10922 & 0.25000 & 0.25000 \\ 0.24669 & 0.18181 & 0.22222 & 0.12500 & 0.18181 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.24669 & 0.09090 & 0.22222 & 0.12500 & 0.18181 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.09790 & 0.18181 & 0.11111 & 0.12500 & 0.18181 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.15540 & 0.18181 & 0.11111 & 0.25000 & 0.09090 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.15540 & 0.18181 & 0.11111 & 0.25000 & 0.18181 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.09790 & 0.18181 & 0.22222 & 0.12500 & 0.18181 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (19)$$

To calculate w and v from (19), we calculate the principle eigen vector corresponding to the eigen value 1.

As a result, normalizing the sum of weights equal to one, we have (20) and (21), as follows:

$$v = \begin{bmatrix} 0.170038 \\ 0.236658 \\ 0.200461 \\ 0.165426 \\ 0.227416 \end{bmatrix}, \quad (20)$$

$$w = \begin{bmatrix} 0.191550 \\ 0.170035 \\ 0.143976 \\ 0.153758 \\ 0.174432 \\ 0.166249 \end{bmatrix}. \qquad (21)$$

Then we have the order of criteria, $c_2 > c_5 > c_3 > c_1 > c_4$. And we have the order of alternatives, $a_1 > a_5 > a_2 > a_6 > a_4 > a_3$.

Next, the results, the order of alternative for various values of θ, are shown in Table 4.

Table 4 The order of alternative for various values of θ

θ	The order of alternative					
2	a_1	a_5	a_2	a_6	a_4	a_3
4	a_1	a_5	a_2	a_6	a_4	a_3
8	a_1	a_5	a_2	a_6	a_4	a_3
16	a_1	a_5	a_6	a_2	a_4	a_3
32	a_1	a_5	a_6	a_2	a_4	a_3
64	a_1	a_5	a_6	a_2	a_4	a_3
128	a_1	a_5	a_6	a_2	a_4	a_3
512	a_1	a_5	a_6	a_2	a_4	a_3

The result of θ=8 or above, in Table 4, a_2 and a_6 reverse the order. However, we do not know suitable value of θ. Moreover the order of criterion and alternatives in ANP and AHP, where θ=2, are summarized in Table 5.

Table 5 The order of criterion and alternatives in ANP and AHP

	The order of criterion					The order of alternative					
ANP	c_2	c_5	c_3	c_1	c_4	a_1	a_5	a_2	a_6	a_4	a_3
AHP (frequent)	c_2	c_5	c_3	c_1	c_4	a_1	a_6	a_5	a_2	a_3	a_4
AHP (infrequent)	c_4	c_1	c_3	c_5	c_2	a_1	a_5	a_2	a_4	a_6	a_3

In Table 5, the result of the order of criterion in ANP coincide with AHP in frequent case, however, the value of weights are not coincide.

4. Conclusion

In this paper, a method to speedily construct comparison matrix and super matrix was proposed and applied to example of AHP and 2-cluster ANP. By preparing each standard of criterion, comparison matrix and super matrix are constructing automatically and have weights immediately. As a result, by our method, it is possible to implement AHP in decision-making part of several software. In future, we need more discussion for accuracy of binary comparisons.

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