Analyses of Pairwise Comparisons with a Ternary Diagram

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ABSTRACT

We provide a method to detect weights of three elements, which can be substitute for the pairwise comparison method in AHP. In the method, decision makers detect the weights by pointing a point on a ternary diagram. We can also analyze ordinary pairwise comparisons with the ternary diagram. We show the following results: (1) the principal eigenvector, which is an output of the pairwise comparison method, is arranged in an inner triangle in the diagram, (2) a new type of contradiction on parametric pairwise comparisons is represented in the diagram, (3) decision makers can check easily a cycle of preference with the diagram, and (4) the area of an inner triangle has very close relationship with the C.I.

Keywords: pairwise comparison, ternary diagram, C.I.

1. Introduction

AHP consists of three steps: building hierarchical structure, evaluation of elements using pairwise comparison method, and synthesis of evaluation values. We propose a new method for the evaluation step. The method enables decision makers to detect evaluation values of three elements by just pointing a point on a ternary diagram, which is a kind of visual scaling methods.

2. Literature Review

Our method is based on convex analysis. A point in a triangle can represent a tuple of weights of three elements. A diagram, which represents the tuple of the weights as a point in a triangle, is referred to as ternary plot. First literature which we found of ternary plot is Gibbs's paper in 1876.

3. Hypotheses/Objectives

Using our method, decision makers detect weights of three elements by pointing a point on a triangle. We assume that three elements can be arranged in the vertex of the triangle.

4. Research Design/Methodology

We consider pointing a point according to a tuple of weights on a triangle such as Fig.1. Vertices $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ of the triangle in Fig.1 correspond to elements $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$. Decision makers put a point close to the vertex $\boldsymbol{X}$ when the element $\boldsymbol{A}$ has large importance. An inner point $\boldsymbol{P}$ is represented in weighted combination of vertex vectors; $\boldsymbol{P} = w_A\boldsymbol{X} + w_B\boldsymbol{Y} + w_C\boldsymbol{Z}$, where w_A, w_B, w_C ($w_A, w_B, w_C > 0$ and $w_A + w_B + w_C = 1$) are weights of elements $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$. They are detected simply by solving a linear equations system. Decision makers can detect weights of three elements without using pairwise comparison methods.

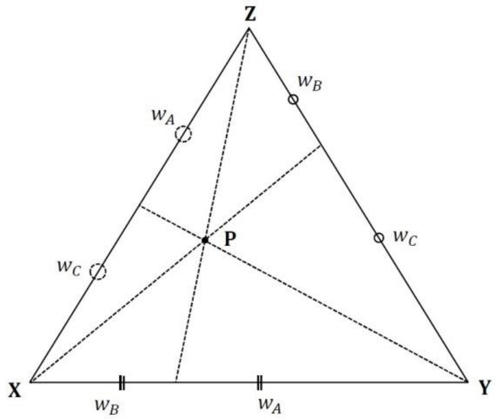

Fig.1

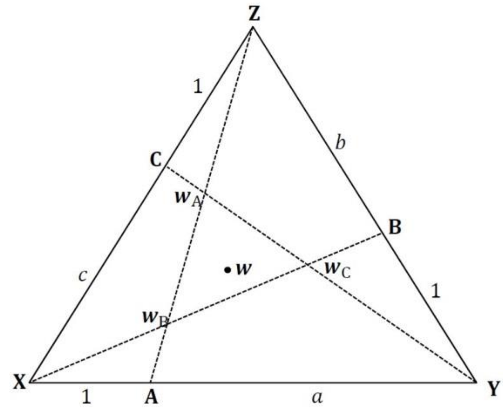

Fig.2

5. Data/Model Analysis

While the method detects weights without inconsistency, we can represent an ordinary pairwise comparison matrix on the diagram (Fig.2); decision makers can analyze a pairwise comparison matrix A of three elements $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ by using the diagram.

$$A = \begin{bmatrix} 1 & a & 1/c \\ 1/a & 1 & b \\ c & 1/b & 1 \end{bmatrix}.$$

(The element $\boldsymbol{A}$ is a times as important as $\boldsymbol{B}$, $\boldsymbol{B}$ is b times as important as $\boldsymbol{C}$, and $\boldsymbol{C}$ is c times as important as $\boldsymbol{A}$)

Analyzing a pairwise comparison matrix on the diagram gives us the results:

(1) The vector which represents the principal eigenvector $\boldsymbol{w}$ of matrix A is in the inner triangle $\varDelta \boldsymbol{w}_A \boldsymbol{w}_B \boldsymbol{w}_C$ in Fig.2.

(2) There is a new type of contradiction of pairwise comparison. If a decision maker compares an element $\boldsymbol{X}$ and an element $\boldsymbol{Y}$, a result of the direct comparison of $\boldsymbol{X}$ and $\boldsymbol{Y}$ can differs from indirect comparison of $\boldsymbol{X}$ and $\boldsymbol{Y}$ via a middle-element $\boldsymbol{Z}$. The decision maker judges whether the contradiction occurs or not by taking a glance at the diagram.

(3) If there is a cycle of preference of elements $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$, then the inner triangle $\varDelta \boldsymbol{w}_A \boldsymbol{w}_B \boldsymbol{w}_C$ contains the center of mass of the outer triangle $\varDelta \boldsymbol{XYZ}$.

(4) The area of triangle $\varDelta \boldsymbol{w}_A \boldsymbol{w}_B \boldsymbol{w}_C$ is $S \frac{1}{1+a^{-1}+c} \frac{1}{a+1+b^{-1}} \frac{1}{c^{-1}+b+1} (abc + (abc)^{-1} - 2)$ where S is the area of the $\varDelta \boldsymbol{XYZ}$ and the term $abc + (abc)^{-1} - 2$ also exists in C.I.

6. Limitations

Our method is for detecting weights of three elements. In pairwise comparisons of n elements ($n > 3$), decision makers pick three from the elements and analyze them by the method. Increasing the number of the elements, however, increases workload of the analyses with increasing the number of combinations of three elements.

7. Conclusions

We provide a method to detect weights of three elements using ternary diagram. And we state that decision makers can analyze ordinary pairwise comparisons with the diagram. Now we develop interfaces for the method.

8. Key References

Gibbs.,J.W. (1876). On the equilibrium of heterogeneous substances, *Transactions of the Connecticut Academy of Arts and Sciences*, 3, pp.108-248, pp.348-520.