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CONCORDANCE, DISCORDANCE, AND SCALING IN THE AHP SUPERMATRIX

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Abstract: A previous paper showed that when concordant the supermatrix undoes the eigenvector normalizations required by AHP, and therefore, that the supermatrix (and hierarchic composition) can be avoided by not normalizing in the first place.
discordance, and shows that while the previous results hold under appropriately scaled concordance under discordance even the supermatrix approach can give arbitrary results.

Introduction

The supermatrix approach is cited by proponents of AHP as the quintessential approach, which under certain circumstances, can be reduced to hierarchic composition (Harker and Vargus, 1987) (Saaty, 1980). A previous paper (Schenkerman, 1994b) showed that under concordance the supermatrix approach achieves its results by undoing the eigenvector normalizations required by AHP. As the conventional weighted-sums approach ensues, the supermatrix (and hierarchic composition) can be avoided by not normalizing in the first place. (This is true also for some proposed variants of AHP. See (Schenkerman, 1994a).)

This paper develops the notion of concordance (supermatrices based upon appropriately scaled objective absolute measurements and some based upon subjective relative measurements), and the notion of discordant supermatrices (all others). It shows that under appropriately scaled concordance the supermatrix approach gives correct results, but under discordance the supermatrix approach can yield arbitrary results.

This paper offers a brief background on supermatrix theory and the theory's main conclusions concerning so-called "independent" criteria weights and overall priorities. Then it provides definitions of and distinctions between concordance and discordance and presents the foregoing assertions as propositions (which are proved in the Appendix). Next it presents a nonpreemptive linear goal program (LGP) for determining concordance/discordance under relative measurements (since under absolute measurements concordance is automatic) and provides examples of both concordance and discordance. It also discusses false concordance and illustrates the effect of unequal LGP objective-function weights under discordance. (Under concordance there is no effect.) Finally, it addresses the need for appropriate scaling of the underlying measurement matrix.

Background on Supermatrices

Let $\mathbf{X} = (x_{ij})$ be an nxm matrix where x_{ij} is the absolute measurement of Alternative i on Criterion j. Without loss of generality, two scalings are assumed. The first scales the columns of $\mathbf{X}$ so the marginal rates of substitution (mrs) between every pair of columns (every pair of criteria) is one. The second scales all x_{ij} so the overall sum, $\Sigma_{ij}x_{ij}$, equals one. (The need for these scalings will be addressed later.) Let $c_j = \Sigma_i x_{ij}$ be the sum of Column j, and $r_i = \Sigma_j x_{ij}$ the sum of Row i (which is valid by the equal-mrs scaling). Let $\mathbf{c}$ be the vector $(\ldots c_j \ldots)'$ and $\mathbf{r}$ be the vector $(\ldots r_i \ldots)'$. By the second scaling above, the components of each vector sum to one.

Let $\mathbf{A} = (a_{ij})$ be the nxm matrix whose elements are $a_{ij} = x_{ij}/c_j$, i.e., are column normalized. Let $\mathbf{B} = (b_{ji})$ be the mxn matrix whose elements are $b_{ji} = x_{ij}/r_i$, i.e., are also column normalized. Since each of their columns sums to one, A and B are column stochastic. (These expressions for a_{ij} and b_{ji} follow from their respective pairwise-comparison matrices, which by virtue of X are consistent.)

A is the matrix of relative scores of the alternatives with respect to the criteria and B the matrix of relative scores of the criteria with respect to the alternatives. Ordinarily, the alternatives are independent of each other as are the criteria. Then, the square supermatrix, W, is

$$W = \left[\begin{array}{c|c} 0 & B \\ \hline A & 0 \end{array}\right].$$

Multiplying W by itself an odd number of times yields

$$W^{2k+1} = \left[\begin{array}{c|c} 0 & B(AB)^k \\ \hline A(BA)^k & 0 \end{array}\right].$$

For k sufficiently large the columns of $B(AB)^k$ become identical as do the columns of $A(BA)^k$. Supermatrix theory holds that each column of $B(AB)^k$ is the vector of independent criteria weights which, if premultiplied by A would yield the overall priorities without resorting to the supermatrix, i.e., would yield the overall priorities by hierarchic composition directly. Supermatrix theory also holds that each column of $A(BA)^k$ is the vector of overall priorities. Note that each column of $(BA)^k$ equals each column of $B(AB)^k$.

Concordance vs Discordance

With absolute measurements in X, the matrices A and B can be derived from X (either directly or from objective pairwise-comparison matrices). In the absence of absolute measurements, i.e., for relative measurements, there is no measurement matrix X and A and B must be developed from pairwise-comparison matrices assessed by the decision maker (DM) subjectively. In the former case it is possible to regenerate X (within a constant of proportionality) from A and B. The regenerated X allows derivation of that same A and that same B. In the latter case, it is possible to compute an induced X from A and B, but the A and B derived from the induced X may not be that used to induce X.

DEFINITION 1. Matrices A and B, and hence the supermatrix comprising them, are concordant if and only if the matrix X they induce yields that same A and that same B. Otherwise, they are discordant. In particular, $a_{ij} = x_{ij}/c_j$ and $b_{ji} = x_{ij}/r_i$ if and only if A and B are concordant.

Thus, concordance implies compatibility (but not cross-determinability) between A and B; discordance, lack of compatibility. It follows that A and B derived from an X determined from absolute measurements will be concordant. Those developed from relative measurements can be discordant. The importance of the distinction lies in the following propositions:

PROPOSITION 1. Under concordance the supermatrix approach works by undoing normalization of the eigenvectors in A. Specifically, each column of $(BA)^k$ and of $B(AB)^k$ is the vector c, the normalized column sums of X; each column of $A(BA)^k$ is the vector of overall priorities, r. Indeed, the supermatrix can be avoided entirely since r can be more easily computed from X directly as the vector of normalized row sums. (See (Schenkerman, 1994b).)

PROPOSITION 2. If A and B are concordant the supermatrix approach gives the correct overall priorities in respect to the underlying X matrix. (Proof in Appendix.)

PROPOSITION 3. *If A and B are discordant the supermatrix approach gives arbitrary overall priorities.* (Proof in Appendix.)

What about the simpler approach known as hierarchic composition? In (Schenkerman, 1994b) it is shown that hierarchic composition is not just a special case of the supermatrix approach, but a very special case. With a concordant supermatrix even hierarchic composition is unnecessary. Indeed, if **X** is known it is not necessary to determine **A** or **B**; the overall priorities can be computed from **X** directly. If **X** is not known, but can be induced from concordant **A** and **B** matrices, the overall priorities can be computed from the induced **X**; indeed, they are computed in the process of determining whether **A** and **B** are concordant or discordant (see next section). Finally, with a discordant supermatrix, the supermatrix approach (and therefore hierarchic composition) can give arbitrary results.

Determining Concordance Under Relative Measurements

As indicated in Proposition 1, under absolute measurements **r**, the overall priorities of the alternatives, can be computed directly from the matrix **X**. There is no need for computing **A** or **B**. Under relative measurements, however, **X** is not directly available--**A** and **B** are elicited from subjective pairwise-comparison matrices. Thus, neither **c** nor **r** is available. How then can the concordance or discordance of **A** and **B** be determined?

Actually, it is relatively simple. A nonpreemptive linear goal program (LGP) can be used. If the objective-function value of the LGP can be made zero, **A** and **B** are concordant; otherwise they are not. In the process, **X** (with both scalings), **c**, and **r** are determined.

Letting ϵ_{ij}^+, ϵ_{ij}^-, δ_{ij}^+, and δ_{ij}^- be nonnegative deviational variables, x_{ij} the elements of the **X** matrix induced from **A** and **B**, and c_j (r_i) the column (row) sums from the induced **X**, the LGP is

$$\text{Min } Z = \Sigma_{ij}(\epsilon_{ij}^+ + \epsilon_{ij}^- + \delta_{ij}^+ + \delta_{ij}^-) \tag{1}$$

subject to

$$a_{ij}c_j - x_{ij} - \epsilon_{ij}^+ + \epsilon_{ij}^- = 0, \text{ all } i, j \tag{2}$$

$$b_{ji}r_i - x_{ij} - \delta_{ij}^+ + \delta_{ij}^- = 0, \text{ all } i, j \tag{3}$$

$$\Sigma_i x_{ij} - c_j = 0, \text{ all } j \tag{4}$$

$$\Sigma_j x_{ij} - r_i = 0, \text{ all } i \tag{5}$$

$$\Sigma_j c_j - \Sigma_i r_i = 0 \tag{6}$$

$$\Sigma_j c_j = 1 \tag{7}$$

all x_{ij}, ϵ_{ij}^+, ϵ_{ij}^-, δ_{ij}^+, δ_{ij}^-, c_j, r_i nonnegative.

Equation (5) forces equal-mrs scaling of the columns (criteria) of the induced **X**. Although (6) is redundant, since it follows from (4) and (5), it clarifies the relationship between column and row sums. Equation (7) scales **c** (and indirectly **r** and **X**) so the elements sum to one.

If Z can be made zero all deviational variables can be made zero and **X** can be induced from **A** and **B** without error. Then **A** and **B** are concordant, **c** is the vector of independent criteria weights, and **r** is the vector of overall priorities.

If Z cannot be made zero some deviational variables remain positive and **X** cannot be induced from **A** and **B** without error. **A** and **B** are discordant. Then, from Proposition 3, the supermatrix approach, and consequently hierarchic composition, are invalid.

Notice that all deviational variables are equally weighted in the LGP's objective function, (1). If Z can be made zero the weights of the deviational variables are immaterial (so long as they are positive). In contrast, if Z cannot be made zero the weights of the deviational variables affect the value of Z. This is not a problem, however, since what matters is the fact that Z and hence all deviational variables cannot be made zero. The actual values of Z and the nonzero deviational variables are irrelevant--the fact of nonzero shows the discordance of **A** and **B**.

Example 1.

As an example of concordant matrices consider the example in (Harker and Vargas, 1987, pp. 1396-1398), also used in (Schenkerman, 1994b, pp. 5-8). The absolute-measurement matrix **X** and the derived **A** and **B** matrices are given in Exhibit 1a. From (Harker and Vargus, 1987), the columns of **X** are scaled for equal marginal rates of substitution, but for simplicity, overall scaling (dividing all **X** entries by the overall sum, 70) has not been done. The **X** matrix has been augmented to show the column, row and overall totals and the normalized column and row sums. As shown in (Harker and Vargus, 1987) and (Schenkerman, 1994b), (both directly and using the supermatrix approach) the normalized row sums, r_i, are the overall priorities and the normalized column sums, c_j, the independent criteria weights.

The LGP using the derived **A** and **B** matrices is shown in the LINDO model of Exhibit M1 (following the Appendix), whose solution is given in Exhibit 1b. Though not shown, the values of the objective function and all deviational variables are zero. Since, as is shown, the induced **X** matrix is the same as the measurement matrix **X** in Exhibit 1a, new **A** and **B** matrices derived from the induced **X** matrix will be the same as those used in the LGP model (the matrices in Exhibit 1a). Thus, **A** and **B** are concordant (as would be expected, being derived from **X**). Therefore, the **c** and **r** vectors are computable using the supermatrix approach, or the LGP, or directly from **X**. (When **X** is available, the last approach is the simplest.)

Example 2.

As an example of discordant matrices consider the example in (Harker and Vargus, 1987, p. 1400) (which is also discussed in (Schenkerman, 1994b, pp. 8-9)). The **A** and **B** matrices and the supermatrix solution from (Harker and Vargus, 1987) are shown in Exhibit 2a. **A** and **B** were not derived from an **X** matrix, but developed from subjective pairwise-comparison matrices. In particular, from (Harker and Vargus, 1987, p. 1400), "... the weights for the three criteria are given exogenously; i.e., these weights are formed independently of the two alternatives." (The statement itself portends that **A** and **B** will be discordant.)

The LINDO model for this example is shown in Exhibit M2 and its solution in Exhibit 2b. The values of the objective function and some deviational variables are positive--proof that **A** and **B** are discordant. Indeed, the independent criteria weights, $(.3, .5, .2)'$, and the overall priorities, $(.5, .5)'$, disconfirm those given by the supermatrix approach.

Exhibit 2c shows the induced **X** matrix and the derived **A** and **B** matrices. Since these derived matrices differ from the original matrices in Exhibit 2a, the original **A** and **B** matrices are discordant.

False Concordance. If the **A** and **B** matrices of Exhibit 2c were used in the LGP (Exhibit M3) the resulting solution (Exhibit 3) would agree with Exhibit 2b, except the values of the objective function and all deviational variables would be zero. This reveals that **A** and **B** matrices derived from even an induced **X** are concordant. In this case, however, this is a false concordance: these are not the **A** and **B** matrices given originally (which presumably were developed from pairwise-comparison matrices assessed by the DM subjectively). Were the derived **A** and **B** matrices the same as those given originally, true concordance would obtain.

Unequal Objective-Function Weights. Finally, to illustrate the arbitrariness of results obtained from discordant **A** and **B** matrices, consider Exhibit M4 which gives the model for Example 2 (Exhibit M2) with unequal objective-function weights. For those deviational variables positive in the solution of Example 2 (Exhibit 2b), the objective-function weights have been increased to 10.

The solution in Exhibit 4 is not the same as in Exhibit 2b, the solution with equal objective-function weights. It can be verified that the **A** and **B** matrices derived from this induced **X** matrix differ from both the original **A** and **B** matrices and from those derived from the **X** induced with equal objective-function weights. Thus, if **A** and **B** are discordant, arbitrary objective-function weights yield arbitrary induced X and derived A and B matrices. In consequence, if A and B are discordant the "overall priorities" and "independent criteria weights" given by the supermatrix approach are arbitrary. In contrast, with concordant A and B matrices the LGP results are independent of the objective-function weights and confirm the results given by the supermatrix approach.

Exhibit 1a X Matrix and Derived A and B Matrices for Example 1

	$Crit_1$	$Crit_2$	$Crit_3$	$Crit_4$	R_i	r_i
Alt_1	1	9	1	3	14	.200000
Alt_2	9	1	9	1	20	.285714
Alt_3	8	1	4	5	18	.257143
Alt_4	4	1	8	5	18	.257143
C_j	22	12	22	14	70	
c_j	.314286	.171429	.314286	.200000		1

$$A = \begin{bmatrix} 1/22 & 9/12 & 1/22 & 3/14 \\ 9/22 & 1/12 & 9/22 & 1/14 \\ 8/22 & 1/12 & 4/22 & 5/14 \\ 4/22 & 1/12 & 8/22 & 5/14 \end{bmatrix} \qquad B = \begin{bmatrix} 1/14 & 9/20 & 8/18 & 4/18 \\ 9/14 & 1/20 & 1/18 & 1/18 \\ 1/14 & 9/20 & 4/18 & 8/18 \\ 3/14 & 1/20 & 5/18 & 5/18 \end{bmatrix}$$

Exhibit 1b LGP Solution for Example 1

Variable	Dec. Value	Fraction
X11	.014286	1/70
X12	.128571	9/70
X13	.014286	1/70
X14	.042857	3/70
X21	.128571	9/70
X22	.014286	1/70
X23	.128571	9/70
X24	.014286	1/70
X31	.114286	8/70
X32	.014286	1/70
X33	.057143	4/70
X34	.071429	5/70
X41	.057143	4/70
X42	.014286	1/70
X43	.114286	8/70
X44	.071429	5/70
C1	.314286	22/70
C2	.171429	12/70
C3	.314286	22/70
C4	.200000	14/70
R1	.200000	14/70
R2	.285714	20/70
R3	.257143	18/70
R4	.257143	18/70

Exhibit 2a A and B Matrices and Supermatrix Solution for Example 2

$$\mathbf{A} = \begin{bmatrix} 0.5 & 0.4 & 0.8 \\ 0.5 & 0.6 & 0.2 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0.3 & 0.3 \\ 0.4 & 0.4 \\ 0.3 & 0.3 \end{bmatrix}$$

$$W^{2k+1} = \begin{bmatrix} 0 & 0 & 0 & 0.3 & 0.3 \\ 0 & 0 & 0 & 0.4 & 0.4 \\ 0 & 0 & 0 & 0.3 & 0.3 \\ 0.55 & 0.55 & 0.55 & 0 & 0 \\ 0.45 & 0.45 & 0.45 & 0 & 0 \end{bmatrix}$$

Exhibit 2b LGP Solution for Example 2

Objective Value = .220000

EP13	.010000
EM23	.010000
DM22	.100000
DP23	.100000
X11	.150000
X12	.200000
X13	.150000
X21	.150000
X22	.300000
X23	.050000
C1	.300000
C2	.500000
C3	.200000
R1	.500000
R2	.500000

Exhibit 2c A and B Matrices from Induced X Matrix

$$\mathbf{X} = \begin{bmatrix} 0.15 & 0.20 & 0.15 \\ 0.15 & 0.30 & 0.05 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0.50 & 0.40 & 0.75 \\ 0.50 & 0.60 & 0.25 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0.30 & 0.30 \\ 0.40 & 0.60 \\ 0.30 & 0.10 \end{bmatrix}$$

Appendix

Proof of Proposition 2

With X known (and equal-mrs scaled), either from absolute measurements or induced from concordant A and B matrices (e.g., using the LGP), **r**, the vector of normalized row sums, is the vector of overall priorities and **c**, the vector of normalized column sums, is the vector of independent criteria weights. Therefore, it is necessary to show that W^{2k+1} gives **r** in the southwest partition and **c** in the northeast partition.

Since, if A and B are concordant (and equal-mrs scaled) $a_{ij}c_j = x_{ij} = b_{ji}r_i$, summing on j yields $\Sigma_j a_{ij}c_j = r_i$ and summing on i yields $\Sigma_i b_{ji}r_i = c_j$. From the definitions of the column vectors **c** and **r** we have

$$\mathbf{Ac} = \mathbf{r} \text{ and } \mathbf{Br} = \mathbf{c}, \tag{A1}$$

where **r** is the stochastic vector of overall priorities and **c** the vector of independent criteria weights. Therefore,

$$\mathbf{BAc} = \mathbf{c} \text{ and } \mathbf{ABr} = \mathbf{r}. \tag{A2}$$

Thus, **c** is the stochastic eigenvector of **BA** and **r** is the stochastic eigenvector of **AB**, both eigenvectors corresponding to the eigenvalue one.

It is well known that starting with a positive vector $\mathbf{Y}^{(0)}$ the eigenvector of **G** corresponding to the eigenvalue one is given by $\mathbf{Y} = \mathbf{G}^k\mathbf{Y}^{(0)}$, for k sufficiently large. Now let $\mathbf{G} \equiv \mathbf{BA}$ and consider the southwest partition of the supermatrix $\mathbf{W}^{2k+1}$. $\mathbf{G}^k$ can be written as $\mathbf{G}^{k-1}\mathbf{G}$. In other words

$$\mathbf{G}^k = [\ldots \mathbf{G}^{(k)}_{.j} \ldots] = \mathbf{G}^{k-1}[\ldots \mathbf{G}_{.j} \ldots]. \tag{A3}$$

Since each column of **G** is positive and stochastic, for k sufficiently large each column of $\mathbf{G}^k$ is the stochastic eigenvector of **G** corresponding to the eigenvalue one; that is, each column of $(\mathbf{BA})^k$ is **c**. Thus, from (A1), each column of $\mathbf{A}(\mathbf{BA})^k$ is $\mathbf{Ac} = \mathbf{r}$, as was to be shown.

Similarly, it can be shown that each column of $(\mathbf{AB})^k$ in the northeast partition of the supermatrix $\mathbf{W}^{2k+1}$ is the stochastic eigenvector **r**. Thus, from (A1), each column of $\mathbf{B}(\mathbf{AB})^k$ is $\mathbf{Br} = \mathbf{c}$, as was to be shown.

Proof of Proposition 3

If A and B are discordant $a_{ij}c_j + \epsilon_{ij} = x_{ij} = b_{ji}r_i + \delta_{ij}$, where c_j, x_{ij}, and r_i are variables and ϵ_{ij} and δ_{ij} are error terms not all of which are zero. Without loss of generality, in addition to equal-mrs scaling, the x_{ij} are assumed scaled so $\Sigma_{ij}x_{ij} = 1$. Then, $\Sigma_j c_j = 1 = \Sigma_i r_i$. Now summing on i yields

$$\Sigma_i a_{ij}c_j + \Sigma_i \epsilon_{ij} = \Sigma_i x_{ij} = c_j = \Sigma_i b_{ji}r_i + \Sigma_i \delta_{ij}, \tag{A4}$$

where, since $\Sigma_i a_{ij} = 1$, $\Sigma_i \epsilon_{ij} = 0$. Summing on j yields

$$\Sigma_j a_{ij}c_j + \Sigma_j \epsilon_{ij} = \Sigma_j x_{ij} = r_i = \Sigma_j b_{ji}r_i + \Sigma_j \delta_{ij}, \tag{A5}$$

where, since $\Sigma_j b_{ji} = 1$, $\Sigma_j \delta_{ij} = 0$. It follows that

$$\mathbf{Ac} + \epsilon\mathbf{1} = \mathbf{r} \text{ and } \mathbf{Br} + \delta'\mathbf{1} = \mathbf{c}, \tag{A6}$$

where $\epsilon = (\epsilon_{ij})$, $\delta = (\delta_{ij})$, and each instance of **1** is a one (sum) vector of appropriate order. Therefore,

$$\mathbf{BAc} + \mathbf{B}\epsilon\mathbf{1} + \delta'\mathbf{1} = \mathbf{c} \text{ and } \mathbf{ABr} + \epsilon\mathbf{1} + \mathbf{A}\delta'\mathbf{1} = \mathbf{r}. \tag{A7}$$

As shown in the Proof for Proposition 2, raising the supermatrix to a sufficiently high odd power, 2k+1, gives the (stochastic) eigenvector of BA, say **q**, for each column of $(\mathbf{BA})^k$ and gives the (stochastic) eigenvector of **AB**, say **p**, for each column of $(\mathbf{AB})^k$ (both **q** and **p** corresponding to the eigenvalue one). This gives **q** for each column in the northeast partition of $\mathbf{W}^{2k+1}$ and **p** for each column in the southwest partition. Thus, **q** and **p** are properties of the supermatrix, while **c** and **r** (which the DM is seeking) are properties of the underlying **X** matrix.

From (A7), **c** is the eigenvector of **BA**, **q**, and **r** is the eigenvector of **AB**, **p**, if and only if

$$\mathbf{B}\epsilon\mathbf{1} + \delta'\mathbf{1} = \mathbf{0} \text{ and } \epsilon\mathbf{1} + \mathbf{A}\delta'\mathbf{1} = \mathbf{0}, \tag{A8}$$

where each instance of **0** is a null vector of suitable order. Of course, (A8) occurs automatically under concordance (where all error terms are zero), but it may also occur under discordance (where some error terms are nonzero). For instance, (A8) occurs if some error terms are nonzero but

$$\epsilon\mathbf{1} = \mathbf{0};\ \delta'\mathbf{1} = \mathbf{0}:\ \Sigma_j\epsilon_{ij} = 0, \text{ all i};\ \Sigma_i\delta_{ji} = 0, \text{ all j}. \tag{A9}$$

Now AHP's basic subjective pairwise-comparison matrices (alternatives on criteria, which yield the **A** matrix, and criteria on alternatives, which yield the **B** matrix) are surrogates for an implied measurement matrix, **X** (within a multiplicative constant). Whereas (A8) holds under concordance because **A** and **B** matrices agree on the unique (see below) underlying **X** matrix, no pair of discordant **A** and **B** matrices agrees on any **X**. Rather, every pair of discordant **A** and **B** matrices estimates (with nonzero error) an infinite number of **X** matrices (differing by more than a scale factor). In particular, as discussed in connection with the discordant LGP, different objective-function weightings yield different **X** matrices and therefore generally different **c** and **r** vectors (as would different objective functions, such as minimizing mean-square error and minimizing maximum error). Yet even though **A** and **B** cannot agree on an underlying measurement matrix, the discordant-supermatrix approach arbitrarily excludes all **X** matrices except those for which (A8) holds, i.e., for which **c** happens to equal **q** and **r** happens to equal **p**. Under discordance the supermatrix approach ignores the existence of all **X** matrices with $\mathbf{c} \neq \mathbf{q}$ and $\mathbf{r} \neq \mathbf{p}$, some of which may provide a lower overall error and be more preferred by the DM than any not excluded. Therefore, as was to be shown, for discordant matrices the supermatrix approach yields arbitrary overall priorities.

Observation

Although the foregoing did not rely on it, the trivial solution, (A9), is the only solution for which (A8) holds. Eliminating $\delta'\mathbf{1}$ from the first equation in (A8) and $\epsilon\mathbf{1}$ from the second yields

$$(\mathbf{AB} - \mathbf{I})\epsilon\mathbf{1} = \mathbf{0} \text{ and } (\mathbf{BA} - \mathbf{I})\delta'\mathbf{1} = \mathbf{0}, \tag{A10}$$

where each instance of **I** is a suitable identity matrix. Thus, for nontrivial solutions to exist for (A8) $\epsilon\mathbf{1}$ must be an eigenvector of **AB** and $\delta'\mathbf{1}$ must be an eigenvector of **BA**. Now, since **A** and **B** are positive matrices, **AB** and **BA** are also positive matrices, and by Perron's Theorem (Saaty, 1980, Theorem 7-4, p. 170) all components of the principal eigenvectors of **AB** and **BA** must be positive. Thus, the sums of each eigenvector, $\mathbf{1}'\epsilon\mathbf{1}$ and $\mathbf{1}'\delta'\mathbf{1}$ must be positive. But, as discussed in connection with (A4) and (A5), $\mathbf{1}'\epsilon = 0$ and $\mathbf{1}'\delta' = 0$. Therefore, since $\mathbf{1}'\epsilon\mathbf{1}$ and $\mathbf{1}'\delta'\mathbf{1}$ are zero, $\epsilon\mathbf{1}$ and $\delta'\mathbf{1}$ cannot be nontrivial in (A10), but must be the trivial solutions in (A9).

Uniqueness of Induced X Matrix Under Concordance

It is to be shown that under concordance the induced **X** matrix is unique to within a scale factor. Recalling that LGP scales the induced **X** so $\Sigma_{ij}x_{ij} = 1$, assume **X** is not unique. Then there exists a similarly scaled matrix $\mathbf{Y} = (y_{ij})$ with column sums g_j and row sums t_i. Since **A** and **B** are concordant, $a_{ij}g_j = y_{ij} = b_{ji}t_i$, which in turn implies

$$\mathbf{Ag} = \mathbf{t} \text{ and } \mathbf{Bt} = \mathbf{g} \text{ so } \mathbf{BAg} = \mathbf{g} \text{ and } \mathbf{ABt} = \mathbf{t}. \tag{A11}$$

As **BA** and **AB** are positive, by Perron's Theorem (Saaty, 1980, Theorem 7-4, p. 170), the principal eigenvectors of **BA** and **AB** are unique (to within a scale factor). Since $\mathbf{1}'\mathbf{g} = \mathbf{1}'\mathbf{t} = 1$, $\mathbf{g} = \mathbf{c}$ and $\mathbf{t} = \mathbf{r}$. Then, $a_{ij}c_j = y_{ij} = b_{ji}r_i$, and $y_{ij} = x_{ij}$, as was to be shown.

Exhibit M1 LINDO Model for Example 1

```
MIN     EP11 + EM11 + EP12 + EM12 + EP13 + EM13 + EP14 + EM14
      + EP21 + EM21 + EP22 + EM22 + EP23 + EM23 + EP24 + EM24
      + EP31 + EM31 + EP32 + EM32 + EP33 + EM33 + EP34 + EM34
      + EP41 + EM41 + EP42 + EM42 + EP43 + EM43 + EP44 + EM44
      + DP11 + DM11 + DP12 + DM12 + DP13 + DM13 + DP14 + DM14
      + DP21 + DM21 + DP22 + DM22 + DP23 + DM23 + DP24 + DM24
      + DP31 + DM31 + DP32 + DM32 + DP33 + DM33 + DP34 + DM34
      + DP41 + DM41 + DP42 + DM42 + DP43 + DM43 + DP44 + DM44
ST
         2) - 22 EP11 + 22 EM11 +   C1 - 22 X11 = 0
         3) - 12 EP12 + 12 EM12 + 9 C2 - 12 X12 = 0
         4) - 22 EP13 + 22 EM13 +   C3 - 22 X13 = 0
         5) - 14 EP14 + 14 EM14 + 3 C4 - 14 X14 = 0
         6) - 22 EP21 + 22 EM21 + 9 C1 - 22 X21 = 0
         7) - 12 EP22 + 12 EM22 +   C2 - 12 X22 = 0
         8) - 22 EP23 + 22 EM23 + 9 C3 - 22 X23 = 0
         9) - 14 EP24 + 14 EM24 +   C4 - 14 X24 = 0
        10) - 22 EP31 + 22 EM31 + 8 C1 - 22 X31 = 0
        11) - 12 EP32 + 12 EM32 +   C2 - 12 X32 = 0
        12) - 22 EP33 + 22 EM33 + 4 C3 - 22 X33 = 0
        13) - 14 EP34 + 14 EM34 + 5 C4 - 14 X34 = 0
        14) - 22 EP41 + 22 EM41 + 4 C1 - 22 X41 = 0
        15) - 12 EP42 + 12 EM42 +   C2 - 12 X42 = 0
        16) - 22 EP43 + 22 EM43 + 8 C3 - 22 X43 = 0
        17) - 14 EP44 + 14 EM44 + 5 C4 - 14 X44 = 0
        18) - 14 DP11 + 14 DM11 - 14 X11 +   R1 = 0
        19) - 14 DP12 + 14 DM12 - 14 X12 + 9 R1 = 0
        20) - 14 DP13 + 14 DM13 - 14 X13 +   R1 = 0
        21) - 14 DP14 + 14 DM14 - 14 X14 + 3 R1 = 0
        22) - 20 DP21 + 20 DM21 - 20 X21 + 9 R2 = 0
        23) - 20 DP22 + 20 DM22 - 20 X22 +   R2 = 0
        24) - 20 DP23 + 20 DM23 - 20 X23 + 9 R2 = 0
        25) - 20 DP24 + 20 DM24 - 20 X24 +   R2 = 0
        26) - 18 DP31 + 18 DM31 - 18 X31 + 8 R3 = 0
        27) - 18 DP32 + 18 DM32 - 18 X32 +   R3 = 0
        28) - 18 DP33 + 18 DM33 - 18 X33 + 4 R3 = 0
        29) - 18 DP34 + 18 DM34 - 18 X34 + 5 R3 = 0
        30) - 18 DP41 + 18 DM41 - 18 X41 + 4 R4 = 0
        31) - 18 DP42 + 18 DM42 - 18 X42 +   R4 = 0
        32) - 18 DP43 + 18 DM43 - 18 X43 + 8 R4 = 0
        33) - 18 DP44 + 18 DM44 - 18 X44 + 5 R4 = 0
        34) - C1 + X11 + X21 + X31 + X41 = 0
        35) - C2 + X12 + X22 + X32 + X42 = 0
        36) - C3 + X13 + X23 + X33 + X43 = 0
        37) - C4 + X14 + X24 + X34 + X44 = 0
        38)   X11 + X12 + X13 + X14 - R1 = 0
        39)   X21 + X22 + X23 + X24 - R2 = 0
        40)   X31 + X32 + X33 + X34 - R3 = 0
        41)   X41 + X42 + X43 + X44 - R4 = 0
        42)   C1 + C2 + C3 + C4 - R1 - R2 - R3 - R4 = 0
        43)   C1 + C2 + C3 + C4 = 1
```

EPij = ϵ^{+}_{ij}, EMij = ϵ^{-}_{ij}, DPij = δ^{+}_{ij}, DMij = δ^{-}_{ij},
Xij = x_{ij}, Cj = c_j, Ri = r_i

Exhibit M2 LINDO Model for Example 2

```
MIN    EM11 + EP11 + EM12 + EP12 + EM13 + EP13
     + EM21 + EP21 + EM22 + EP22 + EM23 + EP23
     + DM11 + DP11 + DM12 + DP12 + DM13 + DP13
     + DM21 + DP21 + DM22 + DP22 + DM23 + DP23
ST
        2)   10 EM11 - 10 EP11 + 5 C1 - 10 X11 = 0
        3)   10 EM12 - 10 EP12 + 4 C2 - 10 X12 = 0
        4)   10 EM13 - 10 EP13 + 8 C3 - 10 X13 = 0
        5)   10 EM21 - 10 EP21 + 5 C1 - 10 X21 = 0
        6)   10 EM22 - 10 EP22 + 6 C2 - 10 X22 = 0
        7)   10 EM23 - 10 EP23 + 2 C3 - 10 X23 = 0
        8)   10 DM11 - 10 DP11 - 10 X11 + 3 R1 = 0
        9)   10 DM12 - 10 DP12 - 10 X12 + 4 R1 = 0
       10)   10 DM13 - 10 DP13 - 10 X13 + 3 R1 = 0
       11)   10 DM21 - 10 DP21 - 10 X21 + 3 R2 = 0
       12)   10 DM22 - 10 DP22 - 10 X22 + 4 R2 = 0
       13)   10 DM23 - 10 DP23 - 10 X23 + 3 R2 = 0
       14)   X11 + X21 - C1 = 0
       15)   X12 + X22 - C2 = 0
       16)   X13 + X23 - C3 = 0
       17)   X11 + X12 + X13 - R1 = 0
       18)   X21 + X22 + X23 - R2 = 0
       19)   C1 + C2 + C3 - R1 - R2 = 0
       20)   C1 + C2 + C3 = 1
```

Exhibit M3 LINDO Model for Derived A and B Matrices of Example 2

```
MIN    EM11 + EP11 + EM12 + EP12 + EM13 + EP13
     + EM21 + EP21 + EM22 + EP22 + EM23 + EP23
     + DM11 + DP11 + DM12 + DP12 + DM13 + DP13
     + DM21 + DP21 + DM22 + DP22 + DM23 + DP23
ST
        2)   10 EM11 - 10 EP11 + 5   C1 - 10 X11 = 0
        3)   10 EM12 - 10 EP12 + 4   C2 - 10 X12 = 0
        4)   10 EM13 - 10 EP13 + 7.5 C3 - 10 X13 = 0
        5)   10 EM21 - 10 EP21 + 5   C1 - 10 X21 = 0
        6)   10 EM22 - 10 EP22 + 6   C2 - 10 X22 = 0
        7)   10 EM23 - 10 EP23 + 2.5 C3 - 10 X23 = 0
        8)   10 DM11 - 10 DP11 - 10 X11 +   3 R1 = 0
        9)   10 DM12 - 10 DP12 - 10 X12 +   4 R1 = 0
       10)   10 DM13 - 10 DP13 - 10 X13 +   3 R1 = 0
       11)   10 DM21 - 10 DP21 - 10 X21 +   3 R2 = 0
       12)   10 DM22 - 10 DP22 - 10 X22 +   6 R2 = 0
       13)   10 DM23 - 10 DP23 - 10 X23 +   1 R2 = 0
       14)   X11 + X21 - C1 = 0
       15)   X12 + X22 - C2 = 0
       16)   X13 + X23 - C3 = 0
       17)   X11 + X12 + X13 - R1 = 0
       18)   X21 + X22 + X23 - R2 = 0
       19)   C1 + C2 + C3 - R1 - R2 = 0
       20)   C1 + C2 + C3 = 1
```

Exhibit M4 LINDO Model for Example 2 with Unequal Weights

```
MIN    EM11 + EP11 +    EM12 + EP12 +    EM13 + 10 EP13
     + EM21 + EP21 +    EM22 + EP22 + 10 EM23 +    EP23
     + DM11 + DP11 +    DM12 + DP12 +    DM13 +    DP13
     + DM21 + DP21 + 10 DM22 + DP22 +    DM23 + 10 DP23
ST
        2)  10 EM11 - 10 EP11 + 5 C1 - 10 X11 = 0
        3)  10 EM12 - 10 EP12 + 4 C2 - 10 X12 = 0
        4)  10 EM13 - 10 EP13 + 8 C3 - 10 X13 = 0
        5)  10 EM21 - 10 EP21 + 5 C1 - 10 X21 = 0
        6)  10 EM22 - 10 EP22 + 6 C2 - 10 X22 = 0
        7)  10 EM23 - 10 EP23 + 2 C3 - 10 X23 = 0
        8)  10 DM11 - 10 DP11 - 10 X11 + 3 R1 = 0
        9)  10 DM12 - 10 DP12 - 10 X12 + 4 R1 = 0
       10)  10 DM13 - 10 DP13 - 10 X13 + 3 R1 = 0
       11)  10 DM21 - 10 DP21 - 10 X21 + 3 R2 = 0
       12)  10 DM22 - 10 DP22 - 10 X22 + 4 R2 = 0
       13)  10 DM23 - 10 DP23 - 10 X23 + 3 R2 = 0
       14)  X11 + X21 - C1 = 0
       15)  X12 + X22 - C2 = 0
       16)  X13 + X23 - C3 = 0
       17)  X11 + X12 + X13 - R1 = 0
       18)  X21 + X22 + X23 - R2 = 0
       19)  C1 + C2 + C3 - R1 - R2 = 0
       20)  C1 + C2 + C3 = 1
```

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