OPTIMUM PRIORITY WEIGHT ESTIMATION METHOD FOR PAIRWISE COMPARISON MATRIX

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ABSTRACT

What weight estimation method to choose in the Analytic Hierarchy Process is an important study subject, because each local estimate weight value depends on an employed weight estimation method and hence the global decision making result of the AHP, which is obtained by integrating estimated local weights, is also affected by the employed weight estimation method. The optimality of the weight estimation method is defined, and on the basis of this optimality concept, a simulation experiment is designed and carried out to statistically find out the optimum weight estimation method in the framework of row-wise generalized mean weight estimation with a parameter *p*. From the simulation experimental result, the application guideline for the optimum weight estimation method is established, classified according to the magnitude of Consistency Index value, the pattern of estimated weight, and the number of comparison items.

Keywords: optimum priority weight, weight estimation method, simulation experiment, generalized mean, pairwise comparison

1. Introduction

Statistical simulation experiment is designed and carried out to find out the optimum weight estimation method in the framework of row-wise generalized mean with a parameter *p* for a pairwise comparison matrix of full information in the AHP. Here, the optimum weight estimation method is defined as the method of estimating the optimum weight vector, and the optimum weight vector is defined in two ways, one as the weight vector which is the closest to the true weight vector, and the other as the weight vector which minimizes the logical inconsistency, or the CI value. From the simulation experimental result, the guideline for applying the optimum weight estimation method is established, classified according to the magnitude of CI value, the pattern of estimated weight vector, and the number of comparison items. The obtained guideline indicates that the row-wise geometric mean method is generally optimum both in the sense of estimating the true weight and in the sense of minimizing the CI value.

2. Related works

Before explaining our simulation experimental design, we will review two of our former related simulation studies ([1] and [2]).

2.1 Miyake's simulation study[1]

Assuming the existence of a true weight vector and adding noise (error term) to each element of ideal pairwise comparison matrix, a simulation of complete pairwise comparison is run. The relative positioning among the true weight vector, the estimated eigenvector, the estimated row-wise geometric mean vector, the estimated row-wise arithmetic mean vector, and the estimated row-wise harmonic mean vector, is obtained. Depending on the type of noise and the magnitude of noise, the estimated weight vector which is the closest to the true weight vector can be guessed from the obtained relative positioning result. But in the actual AHP, since the true weight vector is unknown, or the noise contained in the measured pairwise comparison data cannot be separated, this simulation result cannot teach what weight estimation method is optimum. Here, various error types (multiplicative, additive, etc), various error distributions (uniform, normal, lognormal, etc) and discretization effect of pairwise judgement value (i.e. quantization noise) are taken into consideration, but only four estimated weight vectors (the eigenvector, the row-wise arithmetic mean vector, the row-wise geometric mean vector, and the row-wise harmonic mean vector) are considered, and the only one pattern of the true weight vector $w(*)$, the increasing pattern, is considered ($w(*) = (1, 2, 3, 4, 5)^T$).

2.2 Utsugizaki's simulation study [2]

In order that the simulation result by Miyake be utilized in choosing the optimum weight estimation method in the actual AHP, the relationship between the noise magnitude artificially added in the simulation experiment and the measured CI value is investigated via simulation study. The simulation study result indicates that the noise magnitude and the CI value are strongly correlated with the correlation coefficient around 0.9. From the correspondence between the noise magnitude and the CI value, the table for the optimum weight estimation method is constructed. But also in this table, only small number of weight estimation methods are considered as in Miyake's work (geometric, arithmetic and harmonic means), and hence the simulation result is summarized so that one of the three methods is chosen as the optimum method. Here, the eigenvector is omitted, since the eigenvector is shown to be very close to the row-wise geometric mean vector ([1]).

3. Design principle for proposed simulation experiment

Refining our former simulation experiments ([1] and [2]), an improved version of simulation experiment is designed as follows.

(1) Not only the three weight estimation methods (geometric, arithmetic and harmonic mean), but also a wider range of weight estimation methods should be considered for the candidate of the optimum weight estimation method by introducing the concept of "generalized mean".
(2) For the optimality of the weight estimation method, not only the closeness to the true weight vector but also the smallness of logical inconsistency among pairwise comparisons (or Consistency Index) should be considered, and the relationship between the two optimality concepts be investigated.
(3) The table or the guideline for the optimum weight estimation method should be classified according to the CI value, which is actually measurable in estimating a weight vector by any method, not according to the magnitude of noise, which is artificially added in the simulation experiment and hence is actually not measurable in estimating a weight vector.
(4) Various patterns of true weight vectors should be considered, not only the increasing pattern $w(*) = (1, 2, 3, 4, 5)^T$, but also the all-equal pattern $w(*) = (1, 1, 1, 1, 1)^T$ and others such as $w(*) = (1, 1, 1, 2, 3)^T$, etc.
(5) The number of compared items n should be not only 5 but also 10.
(6) As for the noise artificially added in the simulation, the multiplicative random variable is employed. The random variable is assumed to follow the uniform distribution with the probability density function $[1-\sigma, 1+\sigma]$. Other distributions should be considered such as log-normal distribution.

4. Assumptions and conditions used in the simulation

In the simulation, we assume the unique existence of a true weight vector and the case of complete-information pairwise comparison matrix, and the row-wise generalized mean vector with the parameter p is considered as the candidate of the optimum weight vector, which will be explained in details.

4.1 Unique truth

We assume that there exists a unique true weight vector $w(*)$. Fully-consistent pairwise comparison matrix $W = \{w_{ij}\}$ with its (i, j)th element w_{ij} being given by $w_i(*)/w_j(*)$, is associated with the true weight vector $w(*)$, where $w_i(*)$ is the ith element of vector $w(*)$. We also assume that a pairwise comparison matrix $A = \{a_{ij}\}$ with its (i, j)th element a_{ij} being given by $w_{ij} \times e_{ij}$, is observed, where e_{ij} is the multiplicative error term for the (i, j)th element. That is, the matrix A with some noise added at each element of W is measured.

Following multi-truth cases are not treated in this simulation.

Case 1: Mind is not stable at one point. That is, the person has more than one thought for the true vector.

Case 2: Group of people having different true vectors.

4.2 Complete-information pairwise comparison matrix

We assume that all the elements of pairwise comparison matrix $A = \{a_{ij}\}$ are measured. It is also assumed that $a_{ii} = 1$ (self-identity) and $a_{ij} \cdot a_{ji} = 1$ (reciprocity). The case of incomplete-information pairwise comparison matrix is not treated in this simulation.

4.3 Row-wise generalized mean vector

The row-wise generalized mean vector with the parameter p is employed to estimate the weight vector from the measured pairwise comparison matrix A. The generalized mean with the parameter p for a set of n positive data $a = \{a_1, a_2, \ldots, a_n\}$, $G(p, a)$, is defined by Eq.(1).

$$G(p,a) = \left(\frac{1}{n} \sum_{j=1}^{n} a_j^p \right)^{\frac{1}{p}} \tag{1}$$

The generalized mean is a generalized idea of the mean, which includes the arithmetic mean, the geometric mean, the harmonic mean, and so on. Following properties are known to hold.

Property 1 (Minimum)

$$\lim_{p \to -\infty} G(p,a) = \min_j \{a_j\} \tag{2}$$

Property 2 (Harmonic mean)

$$G(-1,a) = \left(\frac{1}{n} \sum_j a_j^{-1} \right)^{-1} \tag{3}$$

Property 3 (Geometric mean)

$$\lim_{p \to 0} G(p,a) = \left(\prod_j a_j \right)^{\frac{1}{n}} \tag{4}$$

Property 4 (Arithmetic mean)

$$G(1,a) = \frac{1}{n} \sum_j a_j \tag{5}$$

Property 5 (Maximum)

$$\lim_{p \to \infty} G(p,a) = \max_j \{a_j\} \tag{6}$$

Property 6 (Increasing with *p*)

$$\text{If } p_1 < p_2 \text{, then } G(p_1,a) \le G(p_2,a) \tag{7}$$

This generalized mean with the parameter p is applied to the ith row $a_i = \{a_{i1}, a_{i2}, \ldots, a_{in}\}$ of the pairwise comparison matrix $A = \{a_{ij}\}$ to estimate the weight w_i of the ith item.

$$w_i = \left(\frac{1}{n} \sum_{j=1}^{n} a_{ij}^{p} \right)^{\frac{1}{p}} \quad (8)$$

From the properties 1 through 5 and the fact that row-wise geometric mean vector very well approximates the right-eigenvector, the application of the generalized mean with the parameter p (or the pth-order row-wise generalized mean) is expected to cover a wide range of weight vector estimation methods.

5. Optimality of the estimated weight vector

Given a complete-information pairwise comparison matrix $A = \{a_{ij}\}$, what estimated weight vector is optimum, or what weight vector estimation method is optimum? We will give two definitions for the optimality of the estimated weight vector.

5.1 Optimality on the basis of the closeness to the truth

In an actual process of the AHP, the true weight vector is unknown, but in our simulation experiment we assume the unique existence of a true weight vector, such as $w(*) = (1, 2, 3, 4, 5)^T$. Let $w(k)$ be the estimated weight vector obtained by applying the weight vector estimation method "k" to a complete-information pairwise comparison matrix $A = \{a_{ij}\}$. Then, we can measure the distance between the assumed true weight vector $w(*)$ and $w(k)$ by the qth order distance formula(9).

$$\left\| w(*) - w(k) \right\|_q = \left(\sum_{i=1}^{n} \left| w_i(*) - w_i(k) \right|^q \right)^{\frac{1}{q}} \quad (9)$$

Here, $w_i(*)$ and $w_i(k)$ are the ith elements of vectors $w(*)$ and $w(k)$, respectively. The distance defined by Eq.(9) is the absolute distance, or called Manhattan distance, for q=1, and is Euclidean distance for q=2.

[Definition 5.1]
Given a sample pairwise comparison matrix A and a set of weight estimation methods, the truth-optimum estimated weight vector is defined as that which minimizes the distance to the true weight vector, and the truth-optimum weight estimation method is that which estimates the truth-optimum weight vector. □

[Definition 5.2]
Given a set of sample pairwise comparison matrixes and a set of weight estimation methods, the statistically truth-optimum estimated weight vector is defined as that which minimizes the distance to the true weight vector averaged over the set of sample matrixes, and the statistically truth-optimum weight estimation method is that which estimates the statistically truth-optimum weight vector. □

5.2 Optimality on the basis of minimizing the logical inconsistency

Let x be an estimated weight vector for a complete-information pairwise comparison matrix $A = \{a_{ij}\}$, then the estimated full-consistent pairwise comparison matrix $X = \{x_{ij}\}$ is constructed with $x_{ij} = x_i/x_j$, where x_i is the ith element of vector x, and a_{ij} is related to x_{ij} by Eq.(10), where ε_{ij} is called "estimation error".

$$a_{ij} = x_{ij} \times \varepsilon_{ij} \quad (10)$$

Estimation error ε_{ij} is considered to represent a kind of logical inconsistency in estimating a_{ij}. Then, we can define an average logical inconsistency measure by Eq.(11).

$$\textit{Average Logical Inconsistency} = \left(\frac{1}{n(n-1)} \sum_{i \neq j} \varepsilon_{ij} \right) - 1 \tag{11}$$

Next, we will derive a generalized CI formula which uses the generalized concept of the right principal eigenvalue $\lambda_{\max}$ and can be applied to any weight vector, not confined to the right eigenvector. Let x be a weight vector for a pairwise comparison matrix A, not necessarily the right principal eigenvector of A, then generally Eq.(12) or Eq.(13) does not hold. It holds when x is a right eigenvector of A.

$$Ax = \lambda x \tag{12}$$

$$\sum a_{ij} x_j = \lambda x_i \qquad (i=1, \ \dots \ , \ n) \tag{13}$$

Since the equality for the ith row does not hold in Eq.(12) or Eq.(13) for an arbitrary weight vector x, we adjust the value of λ to be λ_i so as to equalize the left hand side and right hand side values of the ith row, as shown by Eq.(14).

$$\sum a_{ij} x_j = \lambda_i x_i \qquad (i=1, \ \dots \ , \ n) \tag{14}$$

Let λ_{ave} be the arithmetic mean of λ_i ($i = 1, \dots , n$).

$$\lambda_{\text{ave}} = \left(\sum_{i=1}^{n} \lambda_i \right) \Big/ n \tag{15}$$

This averaged eigenvalue λ_{ave} is a generalized concept of right principal eigenvalue $\lambda_{\max}$. Just as Saaty's Consistency Index = $(\lambda_{\max}-1)/(n-1)$, Consistency Index CI_{eigen} for an arbitrary weight vector x is defined by using the arithmetically averaged eigenvalue λ_{ave}.

$$\text{CI}_{\text{eigen}} = \frac{\lambda_{\text{ave}} - n}{n-1} \tag{16}$$

Note that this CI_{eigen} is a generalization of Saaty's Consistency Index. When the weight vector x is the right principal eigenvector, CI_{eigen} is equal to Saaty's Consistency Index. Moreover, this is proved in [3] that CI_{eigen} of Eq.(16) is equal to *Average Logical Inconsistency* of Eq.(11) for any weight vector x, which validates the choice of CI_{eigen} defined by Eq.(16) as a measure of Consistency Index.

[Definition 5.3]
Given a sample pairwise comparison matrix A and a set of weight estimation methods, the consistency-optimum estimated weight vector is defined as that which minimizes the logical inconsistency, CI_{eigen} of Eq.(16), and the consistency-optimum weight estimation method is that which estimates the consistency-optimum weight vector. □

[Definition 5.4]
Given a set of sample pairwise comparison matrixes and a set of weight estimation methods, the statistically consistency-optimum estimated weight vector is defined as that which minimizes the logical inconsistency, CI_{eigen} of Eq.(16), averaged over the set of sample matrixes, and the statistically consistency-optimum weight estimation method is that which gives the statistically consistency-optimum weight vector. □

5.3 Truthfulness vs logical consistency
What is the truth (or the reality) in the AHP ([4])? A lot of arguments are still going on about assuming a true (or real) weight vector. But we have assumed in Section 4.1 that there exists a unique

true weight vector $w(*)$, which is given in the simulation experiment but is not known in pairwise comparison measurement of the actual AHP.

Assuming a true weight vector in the simulation experiment may be meaningless for those people who think there is no truth. But if there is no truth, any weight vector which is estimated by any method can be accepted. How can we compare two weight vectors or more which are induced by different methods? Even if the truth is unknown, we think it is necessary to assume the existence of the truth (, or the reality).

For those people who think there are many truths, assuming a unique true weight vector may be also meaningless. In such a case, let one of the many truths, which is of present interest, be under consideration. Then, our simulation with unique truth assumption can be applied.

The truthfulness of an estimated weight vector $w(k)$, or the closeness to the truth, is measured by the distance formula of Eq.(9). On the other hand, the logical consistency among a set of pairwise comparisons can be measured by $\mathrm{CI}_{\mathrm{eigen}}$ of Eq.(16).

How these two measures, the truthfulness and the logical consistency, be related? Logically consistent arguments result in a truthful result? Logically inconsistent arguments do not result in a truthful result? These questions will be answered through the proposed simulation experiment.

6. Simulation experiment

We propose a simulation procedure which clarifies what estimation method is the most optimum in the sense of minimizing the distance to the truth, given a pairwise comparison matrix A and its observed CI value, and what estimation method is the most optimum in the sense of minimizing the logical inconsistency, given a pairwise comparison matrix A. The steps of the proposed simulation experiment will be explained next.

[Step1] A true weight vector $w(*)$ is assumed to be given, such as by $w(*) = (1, 1, 1, 1, 1)^{\mathrm{T}}$, $w(*) = (1, 2, 3, 4, 5)^{\mathrm{T}}$, $w(*) = (1, 1, 2, 3, 4)^{\mathrm{T}}$, etc for n=5, and by $w(*) = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)^{\mathrm{T}}$, $w(*) = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)^{\mathrm{T}}$, $w(*) = (1, 1, 1, 1, 1, 2, 3, 4, 5, 6)^{\mathrm{T}}$, etc for n=10. In the simulation run, all the elements in $w(*)$ is normalized so that their sum is 1, such as $w(*) = (0.2, 0.2, 0.2, 0.2, 0.2)^{\mathrm{T}}$, for $w(*) = (1, 1, 1, 1, 1)^{\mathrm{T}}$, $w(*) = (1/15, 2/15, 1/5, 4/15, 1/3)^{\mathrm{T}}$, for $w(*) = (1, 2, 3, 4, 5)^{\mathrm{T}}$, and so on.

[Step2] The complete-information and full-consistent pairwise comparison matrix $W = \{w_{ij}\}$ is constructed from the assumed true weight vector $w(*)$, where $w_{ij} = w_i(*)/w_j(*)$.

[Step3] An error term e_{ij} which obeys a uniform distribution with range $[-\sigma, \sigma]$ is generated and a sample pairwise comparison matrix $A = \{a_{ij}\}$ is constructed, where $a_{ij} = w_{ij} \times e_{ij}$. Here, e_{ij} is the multiplicative error term, and it can obey another distribution, such as log-normal distribution, etc. Note that the matrix W is fully consistent (or CI = 0), but the matrix A is generally inconsistent (or CI > 0), thus human judgment process of pairwise comparison is simulated.

[Step4] The pth-order row-wise generalized mean is applied to the sample pairwise comparison matrix A constructed in Step3 to estimate the weight vector $w(p) = (w_1(p), w_2(p), \dots , w_n(p))^{\mathrm{T}}$, where $w_i(p)$ is given by Eq.(17).

$$w_i(p) = \left(\frac{1}{n} \sum_{j=1}^{n} a_{ij}^{p} \right)^{\frac{1}{p}} \qquad (i=1, \dots , n) \tag{17}$$

[Step5] Varying the parameter p from -10 to +10 with stepsize 0.1, the distance between $w(*)$ and $w(p)$ is calculated by Eq.(18) with q=1 and generalized Consistency Index $\mathrm{CI}_{\mathrm{eigen}}$ is calculated for $w(p)$ by Eq.(19).

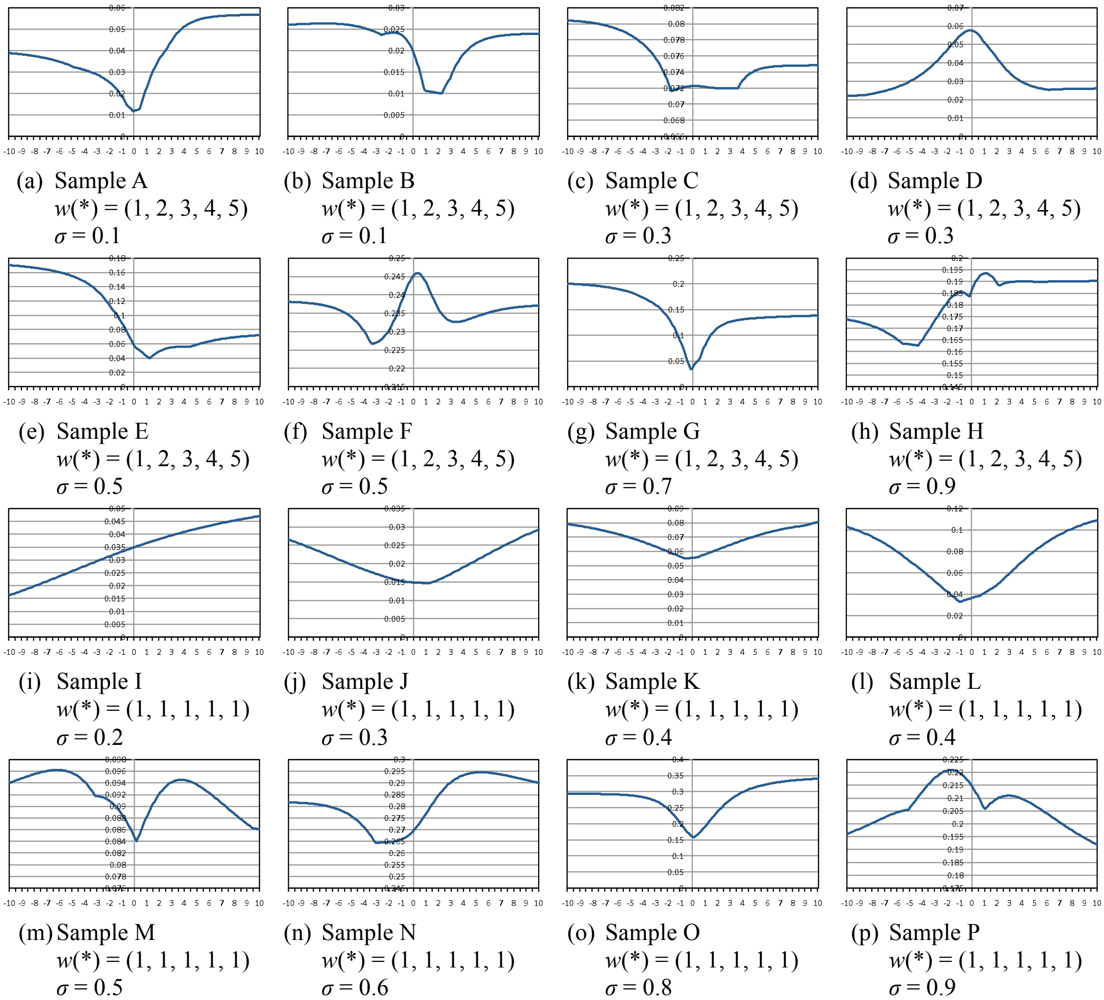
$$\left\| w(*) - w(p) \right\|_q = \left(\sum_{i=1}^{n} \left| w_i(*) - w_i(p) \right|^q \right)^{\frac{1}{q}} \tag{18}$$

$$\mathrm{CI}_{\mathrm{eigen}}(p) = \frac{\lambda_{\mathrm{ave}} - n}{n-1} \tag{19}$$

7. Simulation results

7.1 Distance characteristics of individual samples

The distance characteristics to the true weight vector with the parameter p are shown in Fig.1 for 16 individual samples.

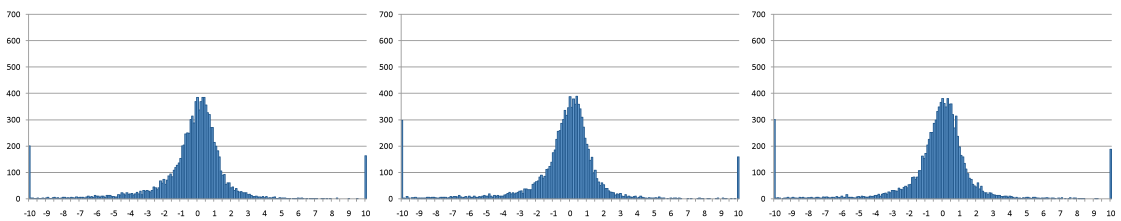


(a) Sample A $w(*) = (1, 2, 3, 4, 5)$ $\sigma = 0.1$
(b) Sample B $w(*) = (1, 2, 3, 4, 5)$ $\sigma = 0.1$
(c) Sample C $w(*) = (1, 2, 3, 4, 5)$ $\sigma = 0.3$
(d) Sample D $w(*) = (1, 2, 3, 4, 5)$ $\sigma = 0.3$
(e) Sample E $w(*) = (1, 2, 3, 4, 5)$ $\sigma = 0.5$
(f) Sample F $w(*) = (1, 2, 3, 4, 5)$ $\sigma = 0.5$
(g) Sample G $w(*) = (1, 2, 3, 4, 5)$ $\sigma = 0.7$
(h) Sample H $w(*) = (1, 2, 3, 4, 5)$ $\sigma = 0.9$
(i) Sample I $w(*) = (1, 1, 1, 1, 1)$ $\sigma = 0.2$
(j) Sample J $w(*) = (1, 1, 1, 1, 1)$ $\sigma = 0.3$
(k) Sample K $w(*) = (1, 1, 1, 1, 1)$ $\sigma = 0.4$
(l) Sample L $w(*) = (1, 1, 1, 1, 1)$ $\sigma = 0.4$
(m) Sample M $w(*) = (1, 1, 1, 1, 1)$ $\sigma = 0.5$
(n) Sample N $w(*) = (1, 1, 1, 1, 1)$ $\sigma = 0.6$
(o) Sample O $w(*) = (1, 1, 1, 1, 1)$ $\sigma = 0.8$
(p) Sample P $w(*) = (1, 1, 1, 1, 1)$ $\sigma = 0.9$

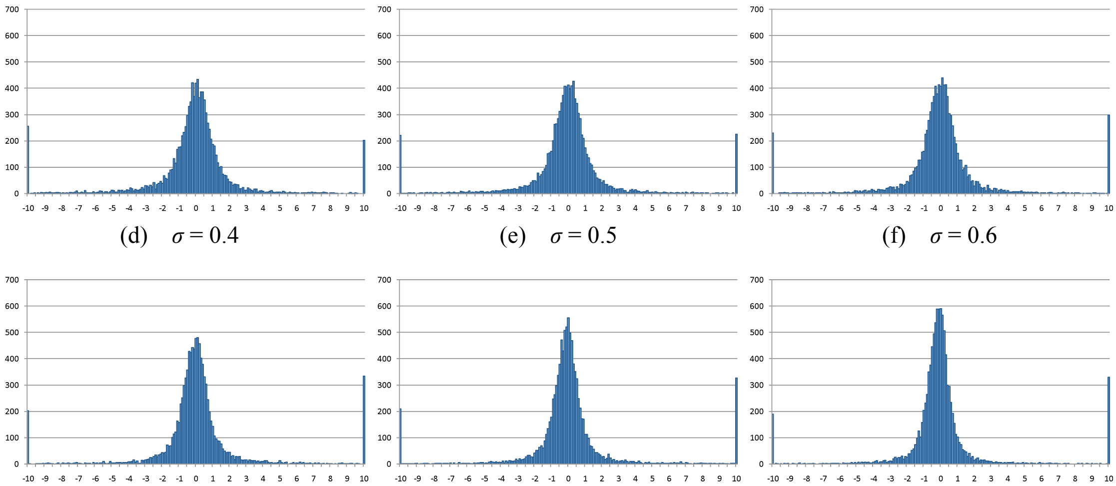
Fig. 1 Distance characteristics for 16 samples, where the horizontal axis shows the parameter p and the vertical axis shows the distance between the estimate and the truth.

7.2 Frequency distribution of minimum distance achievement

The frequency distributions of the parameter p which achieves the minimum distance are shown in Fig.2 for $w(*) = (1, 2, 3, 4, 5)^T$, where the error magnitude σ is set to be 0.1, 0.2, … , 0.9, and its frequency distribution is summarized with σ as the 3rd axis, as shown in Fig.3. The sample size is 10,000 for all the frequency distributions in Fig.2.



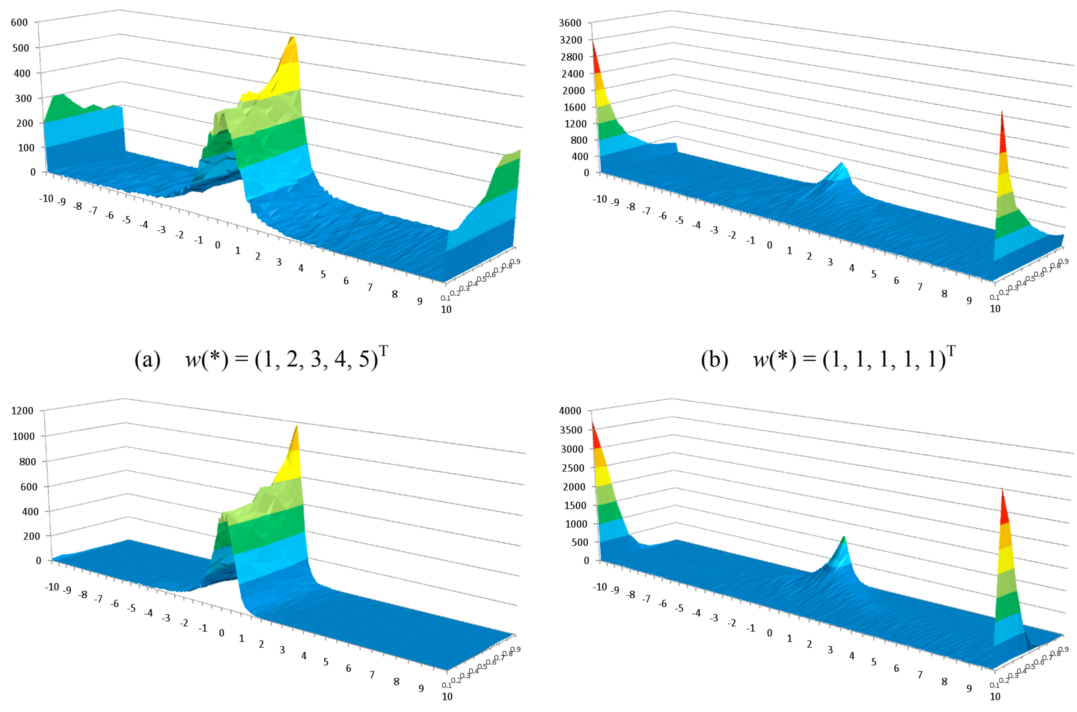
(a) $\sigma = 0.1$ (b) $\sigma = 0.2$ (c) $\sigma = 0.3$



(d) $\sigma = 0.4$ (e) $\sigma = 0.5$ (f) $\sigma = 0.6$

(g) $\sigma = 0.7$ (h) $\sigma = 0.8$ (i) $\sigma = 0.9$

Fig. 2 Frequency distribution of the minimum distance achievement for various values of error magnitude σ with sample size = 10,000 ($w(*) = (1, 2, 3, 4, 5)^T$), where the horizontal axis shows the parameter p and the vertical axis shows the frequency of the minimum distance achievement.



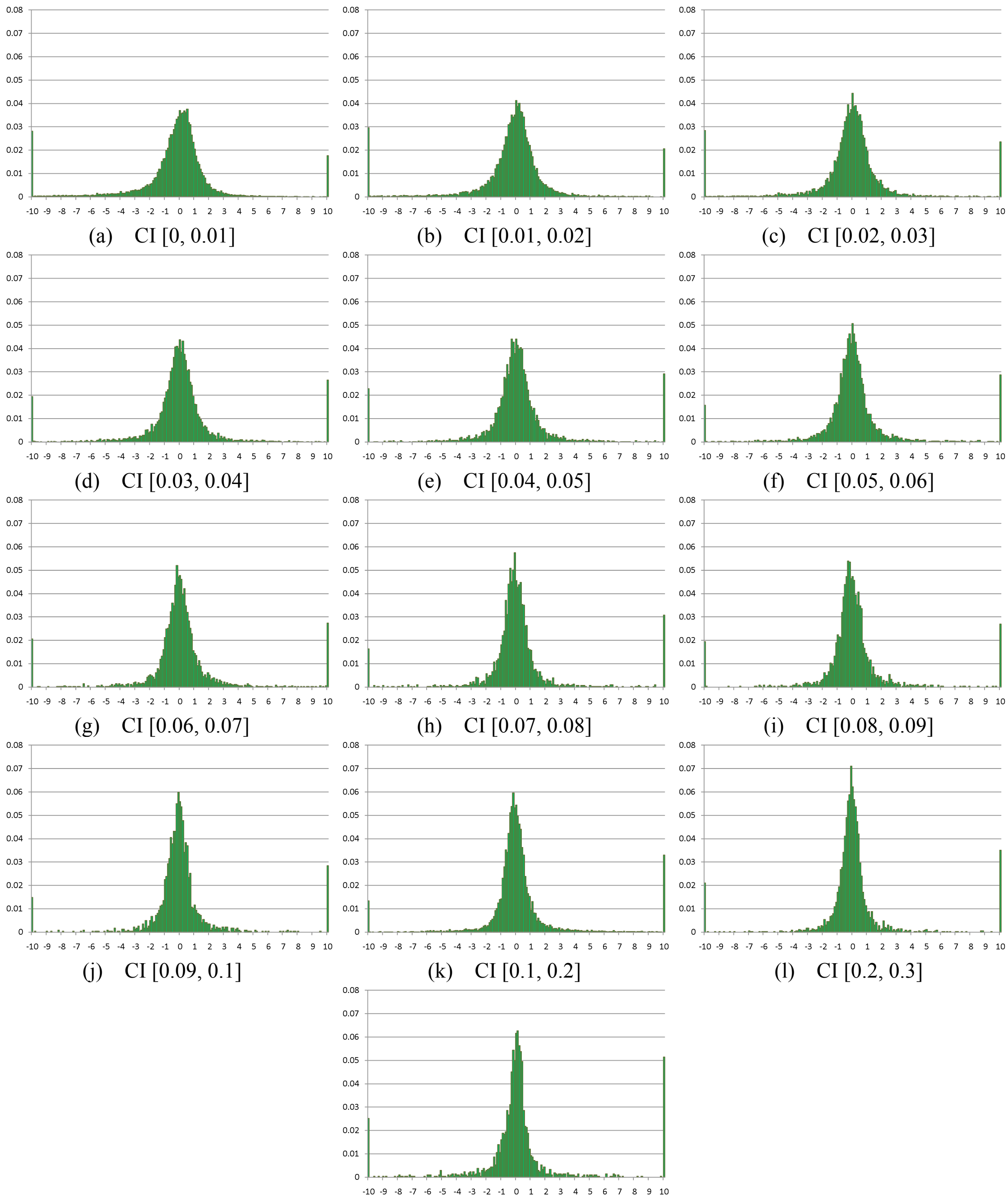
(a) $w(*) = (1, 2, 3, 4, 5)^T$ (b) $w(*) = (1, 1, 1, 1, 1)^T$

(c) $w(*) = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)^T$ (d) $w(*) = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)^T$

Fig. 3 Frequency distribution of the minimum distance achievement with σ as the 3rd axis

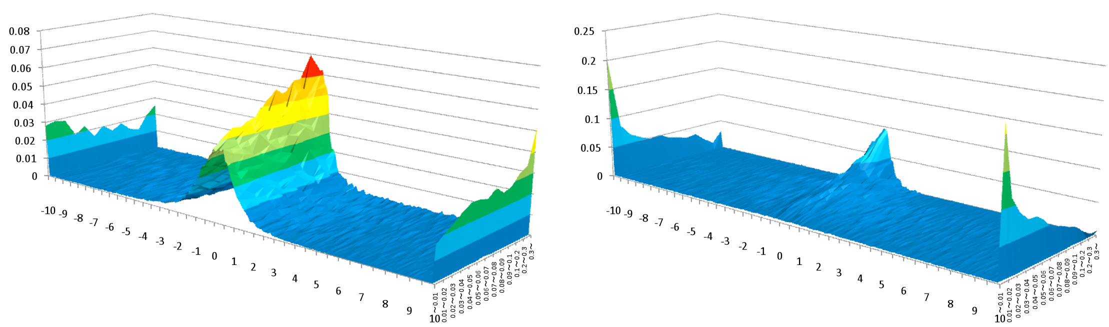
7.3 Frequency distribution of the minimum distance achievement classified by CI ranges

The data of the frequency distributions in Section 7.2 are rearranged so that the data are classified according to CI values, more specifically CI value ranges such as (0.1, 0.2), the range of CI values with $0.1 < \mathrm{CI} \leq 0.2$. Fig.4 shows the frequency distribution of the parameter p which achieves the minimum distance for $w(*) = (1, 2, 3, 4, 5)^T$, where the data are reclassified for thirteen CI ranges, [0, 0.01], (0.01, 0.02], (0.02, 0.03], … , (0.1, 0.2], (0.2, 0.3], and (0.3, ∞). Fig.5 shows the frequency distribution of the minimum distance achievement with CI value as the 3rd axis.

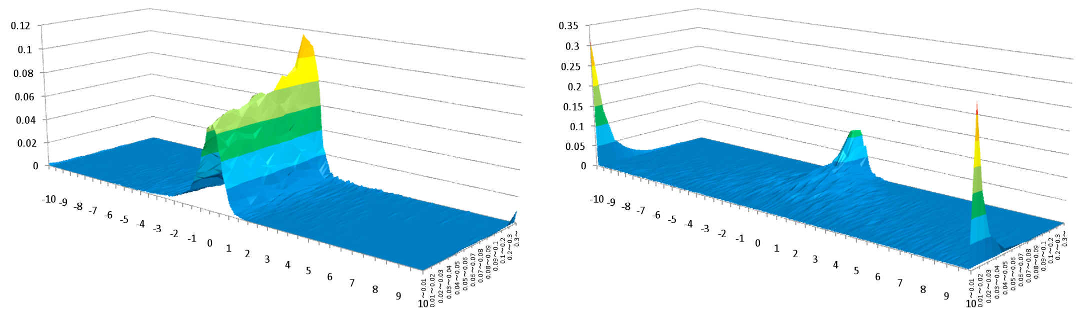


(a) CI [0, 0.01] (b) CI [0.01, 0.02] (c) CI [0.02, 0.03]
(d) CI [0.03, 0.04] (e) CI [0.04, 0.05] (f) CI [0.05, 0.06]
(g) CI [0.06, 0.07] (h) CI [0.07, 0.08] (i) CI [0.08, 0.09]
(j) CI [0.09, 0.1] (k) CI [0.1, 0.2] (l) CI [0.2, 0.3]
(m) CI [0.3, ∞)

Fig. 4 Frequency distribution of the minimum distance achievement for various values of CI range



(a) $w(*) = (1, 2, 3, 4, 5)^T$ (b) $w(*) = (1, 1, 1, 1, 1)^T$

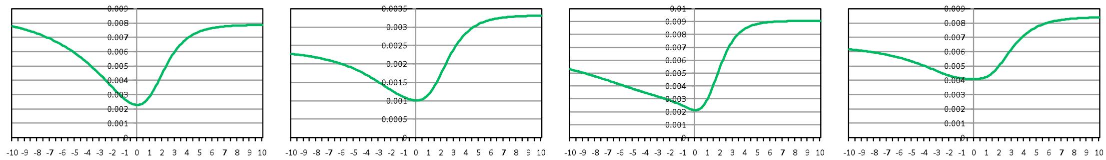


(c) $w(*) = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10)^T$ (d) $w(*) = (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)^T$

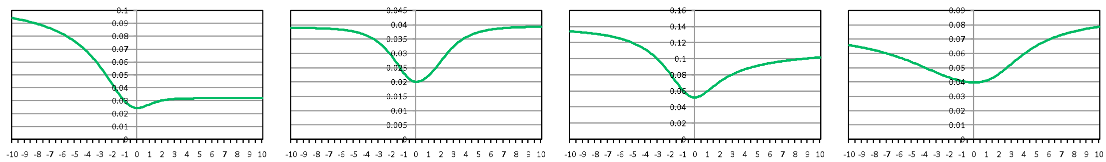
Fig. 5 Frequency distribution of the minimum distance achievement with CI value as the 3rd axis

7.4 CI characteristics of individual samples

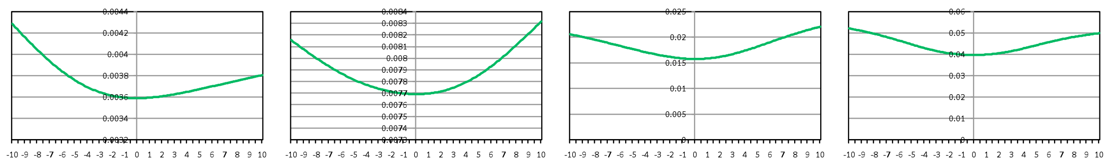
CI value characteristics with the parameter p are shown in Fig.6 for the same 16 individual samples listed in Fig.1.



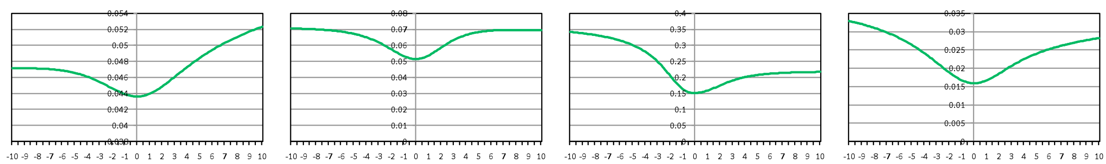
(a) Sample A $w(*) = (1, 2, 3, 4, 5)$ $\sigma = 0.1$
(b) Sample B $w(*) = (1, 2, 3, 4, 5)$ $\sigma = 0.1$
(c) Sample C $w(*) = (1, 2, 3, 4, 5)$ $\sigma = 0.3$
(d) Sample D $w(*) = (1, 2, 3, 4, 5)$ $\sigma = 0.3$



(e) Sample E $w(*) = (1, 2, 3, 4, 5)$ $\sigma = 0.5$
(f) Sample F $w(*) = (1, 2, 3, 4, 5)$ $\sigma = 0.5$
(g) Sample G $w(*) = (1, 2, 3, 4, 5)$ $\sigma = 0.7$
(h) Sample H $w(*) = (1, 2, 3, 4, 5)$ $\sigma = 0.9$



(i) Sample I $w(*) = (1, 1, 1, 1, 1)$ $\sigma = 0.2$
(j) Sample J $w(*) = (1, 1, 1, 1, 1)$ $\sigma = 0.3$
(k) Sample K $w(*) = (1, 1, 1, 1, 1)$ $\sigma = 0.4$
(l) Sample L $w(*) = (1, 1, 1, 1, 1)$ $\sigma = 0.4$



(m) Sample M $w(*) = (1, 1, 1, 1, 1)$ $\sigma = 0.5$
(n) Sample N $w(*) = (1, 1, 1, 1, 1)$ $\sigma = 0.6$
(o) Sample O $w(*) = (1, 1, 1, 1, 1)$ $\sigma = 0.8$
(p) Sample P $w(*) = (1, 1, 1, 1, 1)$ $\sigma = 0.9$

Fig. 6 CI characteristics for the same 16 samples, where the horizontal axis shows the parameter p and the vertical axis shows the CI value (CI_{eigen} of Eq.(16)).

7.5 Frequency distribution of the minimum CI achievement

The frequency distributions of the parameter p which achieves the minimum CI are shown in Fig.7 for $w(*) = (1, 2, 3, 4, 5)^T$, where the error magnitude σ is set to be 0.1, … , 0.9, and these frequency distributions are summarized with σ as the 3rd axis, as shown in Fig.8.

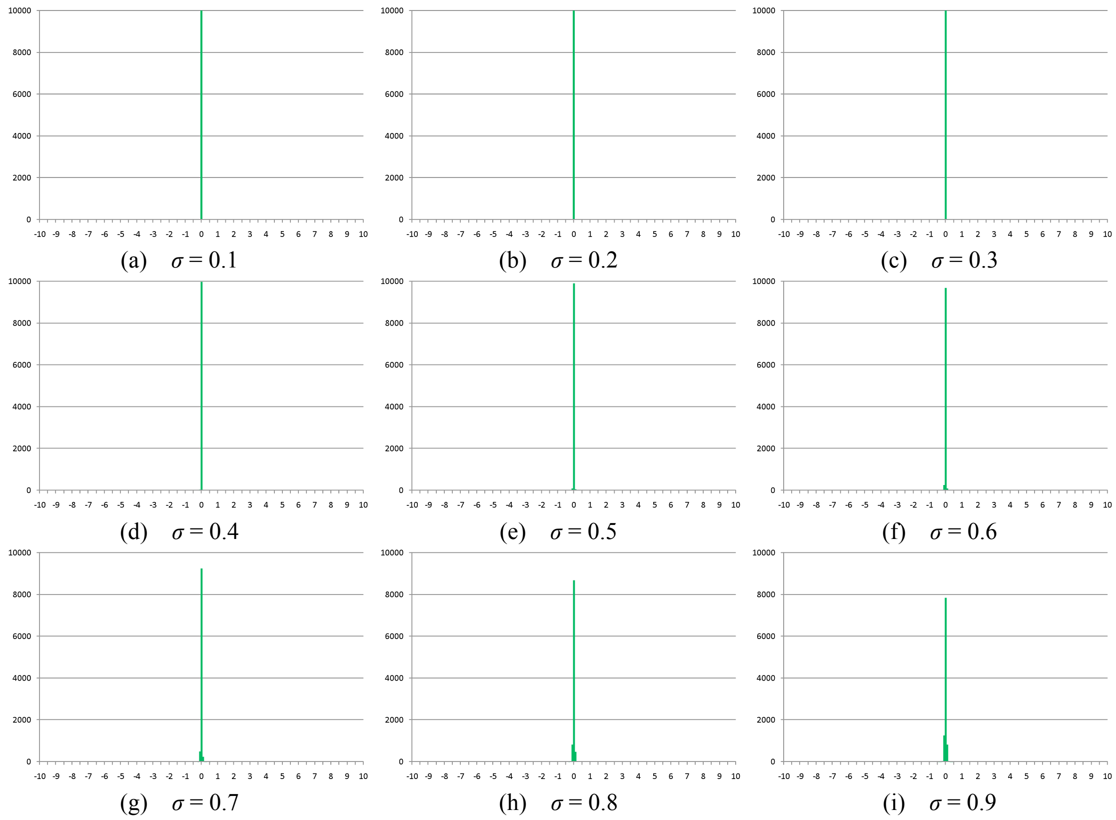


Fig. 7 Frequency distribution of the minimum CI achievement for various values of error magnitude σ($w(*) = (1, 2, 3, 4, 5)^T$)

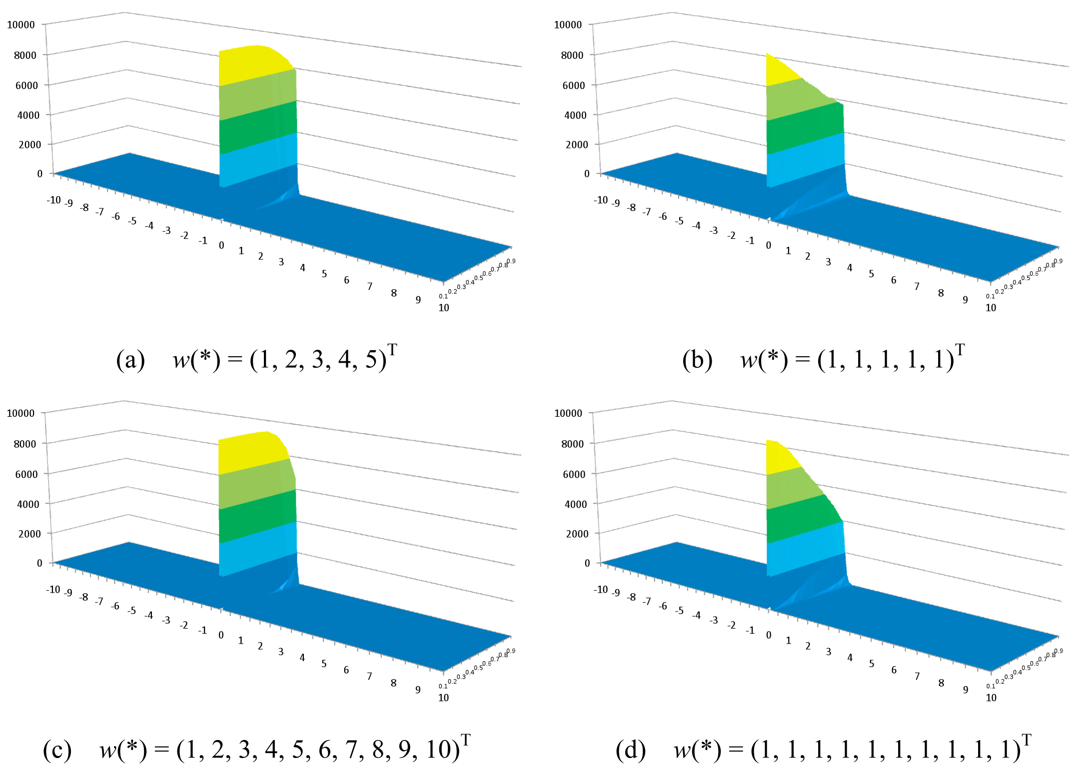


Fig. 8 Frequency distribution of the minimum CI achievement with σ as the 3rd axis

8. Considerations on simulation results

(i) The distance characteristic of each sample differs from one sample to another, as shown in Fig.1. But most of the distance characteristic curves are V-shaped with one local minimum point. Sample F in Fig.1 is an exception, where it has two local minimum points.

(ii) The frequency distribution characteristics of the minimum distance achievement of Figs.2 and 3 indicate that when the noise is very small, or the error magnitude σ is around 0.1 or 0.2, the minimum distance to the truth is achieved relatively uniformly over a very wide range of the parameter value p, regardless of the pattern of the true weight vector, increasing or all-equal type, and the number of compared items, n=5 or n=10. More precisely, this tendency is remarkable for the case of all-equal pattern. As the error magnitude σ is increased, the minimum distance to the truth is achieved more frequently at around the parameter value $p = 0$ (geometric mean).

(iii) The frequency distribution characteristics of the minimum distance achievement, classified by CI values, of Figs.4 and 5 indicate that when the measured CI value is very small, such as CI < 0.01, the minimum distance to the truth is achieved relatively uniformly over a very wide range of the parameter value p, or the parameter value p which yields the minimum distance is uniformly distributed over a very wide range of CI values. More precisely, this tendency is remarkable for the case of all-equal pattern. From another viewpoint, the distance characteristic when the CI value (or the error magnitude σ) is very very small (or zero), the distance characteristic curve is expected to be flat with zero distance for any parameter value p. On the contrary, as the CI value, measured by any weight estimation method with CI_{eigen} of Eq.(16), is increased, the minimum distance to the truth is achieve more frequently at around the parameter value $p = 0$ (geometric mean). The kurtosis (or the sharpness of the peak) of the frequency distribution at $p = 0$ also increases with CI value.

(iv) Comparing Fig.2 and Fig.4, we can observe that the sharpness of the peak for the histograms of Fig.4 is stronger than that of Fig.2. This indicates that the index of measurable CI value is more sensitive in choosing the optimum weight estimation method than the index of unmeasurable error magnitude.

(v) The CI characteristic of each sample differs from one sample to another, as shown in Fig.7. But their differences are small compared to those of the distance characteristic of Fig.1. Note that the minimum CI is achieved at $p = 0$ for all the 16 samples.

(vi) The frequency distribution characteristics of the minimum CI achievement of Figs.7 and 8 indicate that when the error magnitude σ is small, the minimum CI is achieved 100% at the parameter value $p = 0$ (geometric mean), regardless of the pattern of the true weight vector, increasing or all-equal type, and the number of compared items, n=5 or n=10. Although this tendency is strongly maintained as the error magnitude σ is increased, some small number of samples which achieve the minimum CI other than at $p = 0$ can be observed when $\sigma > 0.5$.

(vii) The frequency distribution characteristics of the minimum CI achievement of Figs.7 and 8 are sharp compared to those of the minimum distance achievement of Figs.2-5. This indicates that most of the truthful estimate weights do not necessarily yield the minimum CI.

9. Conclusion

A simulation experiment is designed and carried out to find out the optimum method of weight estimation from the two optimality viewpoints, the truthfulness of estimated weight vector and the logical consistency of estimated weight vector.

When measured CI value is very small, less than 0.01, a wide range of row-wise generalized mean with any parameter value p can estimate a near-optimum weight vector in the sense of minimizing the distance to the truth, especially for the case where estimated weight vector is of all-equal type. As measured CI value is increased, the probability that the row-wise geometric mean ($p = 0$) estimates the optimum weight vector is increased. When CI value is very small, since any generalized mean method with any p estimates a near-optimum weight vector, the geometric mean method also estimates a near-optimum weight vector. From the viewpoint of estimating truthful weight vector it is summarized that, the row-wise geometric mean method ($p = 0$) statistically yields an optimum, or even near-optimum, weight vector, over a wide range of CI values. Moreover, the row-wise geometric mean method (p = 0), with very high probability, minimizes the logical inconsistency, or maximizes the logical consistency.

Since it is known that the eigenvector weight and the row-wise geometric mean weight very well approximate each other (for example see [1]), it is concluded that the row-wise geometric mean method, or namely, the right eigenvector method, is statistically optimum from the viewpoints of both the truthfulness (Definition 5.2) and the logical consistency (Definition 5.4).

Using available measured information, such as CI value, what weight estimation method to choose, not statistically, but corresponding to each individual sample (Definition 5.1 and Definition 5.3), still remains a problem. The relationship between the individual distance characteristic of Fig.1 and the individual CI characteristic of Fig.6, and the extension of our proposed approach to the incomplete-information of comparison matrix, are also future research subjects.

REFERENCES

[1] Chikako Miyake: Simulation Study on AHP Weight Estimation (in Japanese), Master Thesis of Mathematical Engineering, Graduate School of Industrial Technology, Nihon University (2001.3)

[2] Masayuki Utsugizaki: Study on Consistency Index in AHP (in Japanese), Master Thesis of Mathematical Information Engineering, Graduate School of Industrial Technology, Nihon University (2008.3)

[3] Masayuki Utsugizaki, Keikichi Osawa and Masaaki Shinohara: Equivalence between CI values defined by the row-wise eigenvalue and the element-wise error mean (in Japanese), Proceedings of 40th Mathematical Information Research Conference, College of Industrial Technology, Nihon University, pp.79-82 (2007.12.1)

[4] Masaaki Shinohara, Keikichi Osawa, Kazuya Inamine and Itaru Goto: What is the true weight or the truth in psycho-physical experiment? (in Japanese), Proceedings of 38th Mathematical Information Research Conference, College of Industrial Technology, Nihon University, pp.97-98 (2005.12.3)