VOTING WITH INTENSITY OF PREFERENCES

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ABSTRACT

In this paper we develop a method based on the idea of pairwise voting to rank projects or candidates and incorporate in the ranking process how strongly the referees/voters feel about the comparisons they make. Voting is a modified form of ranking and all the votes are equally important. However, there are situations similar to voting in which the votes are not just ordinal but each voter expresses an intensity of preference for the different candidates. For example, ranking projects for funding. We show that our method yields the same results as ordinal voting when the intensity of preferences tends to infinity.

KEYWORDS: voting, group decision making, pairwise comparisons, reciprocal matrices, Analytic Hierarchy Process

1. Introduction

One of the purposes of the electoral process is to elicit and measure the preference of voters. However, the use of voters' preferences is not limited to the election of government officials or board members in the public and private sector. The voting process is also used to rank projects, e.g., in the National Science Foundation or the National Institutes of Health (NIH), and in general, to rank alternative courses of action. In many cases, ranking from the voters' preferences needs to be translated into priorities that can be used to allocate resources, such as prioritizing spending in a portfolio of research projects, when not all the projects can be funded due to budgetary constraints. In general, NIH does not have all the necessary funds to support all the projects requesting funding. Referees then are required to rank the projects making intensity of preference statements according to some pre-established dimensions, such as innovation, institution, researcher and so on, that will yield a cardinal ranking of the projects, i.e., the projects are assigned a score that help the referees decide how to fund them.

In this paper we develop a method based on the idea of pairwise voting to rank projects or candidates and incorporate in the ranking process how strongly the referees/voters feel about the comparisons they make. Although this is not a new problem as noted by (Cook et al., 2005) and (Hochbaum and Levin, 2006), our approach is new.

1.1. Notation and terminology

Let $\mathsf{A} = \{a_1,...,a_m\}$ be a set of alternatives. Let $\mathrm{N} = \{1,2,3,...\}$ be the set of voters. A *preference order* $\sigma = \{a_{i_1},...,a_{i_m}\}$ is a ranking of the alternatives. Let $L(\mathsf{A})$ be the set of all m! orders. A *profile* on a group of voters $M \underset{\neq}{\subset} \mathrm{N}$ is a mapping $\phi : M \to L(\mathfrak{A})$. Let Φ be the set of all the possible profiles. For $\sigma \in L(\mathsf{A})$ and $\phi \in \Phi$ let $n_\sigma(\phi)$ be the number of voters in the profile ϕ that have the preference order σ. A *preference function* is a mapping from the set of profiles Φ to the set of preference orders, $f : \Phi \to L(\mathsf{A})$. A *choice function* is a mapping from the set of profiles Φ to the set of nonempty subsets of A. Let $\left(\sigma_1(\phi),...,\sigma_H(\phi)\right)$ be the different orderings in the profile ϕ. Let $v_{ij}\left[\sigma_h(\phi)\right]$ be the number of voters who prefer i to j in the ordering $\sigma_h(\phi)$ of profile ϕ. Let $n_{\sigma_h}(\phi)$ be number of votes in the ordering $\sigma_h(\phi)$ of profile ϕ. Let $w_i\left[\sigma_h(\phi)\right]$ be the weight assigned to the ith alternative in the ordering $\sigma_h(\phi)$ of profile ϕ.

2. The Eigenvector Method For Pairwise Voting

When people vote in favor of a candidate A versus a candidate B, they basically rank them and state that for example A is preferred to B, i.e., $A \succ B$. If a population of N individuals vote to select between A and B, if n_1 vote for A and n_2 vote for B, then out of the total number of votes $n_1 + n_2$, $(n_1 / n_1 + n_2) \times 100$ percent have voted for A and the remainder for B. Let us assume without loss of generality that $n_1 \geq n_2$. So, for the population of N individuals how much strongly is candidate A preferred over candidate B? The answer is n_1 / n_2 if $n_2 > 0$. The matrix of pairwise comparisons representing the relative strength of preference for the candidates is given by

$$\begin{array}{cc} \text{Voting Matrix} & \text{Weights} \\ \begin{pmatrix} 1 & \frac{n_1}{n_2} \\ \frac{n_2}{n_1} & 1 \end{pmatrix} & \begin{pmatrix} \frac{n_1}{n_1+n_2} \\ \frac{n_2}{n_1+n_2} \end{pmatrix} \end{array}$$

The voting matrix is a positive reciprocal matrix, i.e., if the entries of the matrix are denoted by a_{ij}, then $a_{ji} = a_{ij}^{-1}$ for all i and j. Note that $a_{ij} \equiv \dfrac{n_i}{n_j}$. If instead of two candidates there are m candidates, then because people only vote for one candidate, the voting matrix would be given by

$$W = \begin{pmatrix} 1 & n_1/n_2 & \cdots & n_1/n_m \\ n_2/n_1 & 1 & \cdots & n_2/n_m \\ \vdots & \vdots & \ddots & \vdots \\ n_m/n_1 & n_m/n_2 & \cdots & 1 \end{pmatrix} \tag{1.1}$$

Multiplying this matrix by the column vector $(n_1, n_2, ..., n_m)$ we get

$$\begin{pmatrix} 1 & n_1/n_2 & \cdots & n_1/n_m \\ n_2/n_1 & 1 & \cdots & n_2/n_m \\ \vdots & \vdots & \ddots & \vdots \\ n_m/n_1 & n_m/n_2 & \cdots & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_m \end{pmatrix} = m \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_m \end{pmatrix} \tag{1.2}$$

Since the trace of the matrix is equal to *m*, the trace of a matrix is equal to the sum of all its eigenvalues, and the rank of the *W* is equal to 1, we conclude that *m* must be an eigenvalue of *W*, and it is its largest eigenvalue in modulus. By Perron-Froebenius' theorem (Gantmacher, 1960) it is known that the largest (principal) eigenvalue of a nonnegative real matrix always exists and it is positive and real. In our case, that eigenvalue is equal to *m*. The right eigenvector associated with the principal eigenvalue is unique to within a multiplicative constant and it is given by $\left(\frac{n_1}{N}, \frac{n_2}{N}, \cdots, \frac{n_m}{N}\right)^T$, where *N* is the total number of voters. The principal right eigenvector yields the correct percentage of votes for each of the candidates.

This model assumes that people vote for one candidate only. However, when a person can vote for more than one candidate the matrix of paired comparisons may be inconsistent (Saaty, 1986), i.e., the entries do not satisfy the condition $a_{ij}a_{jk} = a_{ik}$ for all *i*, *j* and *k*. In this paper we are concerned not just with people voting for multiple candidates but ranking them in term of preference.

Recall that we denoted by $v_{ij}(\phi)$ the number of voters in a profile ϕ that prefer candidate *i* to candidate *j*. Let $a_{ij}(\phi) \equiv \frac{v_{ij}(\phi)}{v_{ji}(\phi)}$, $v_{ji}(\phi) > 0$. Note that even if the profile ϕ contains all m! preference orders, the voting matrix $A(\phi)$ whose entries are given by $a_{ij}(\phi) \equiv \frac{v_{ij}(\phi)}{v_{ji}(\phi)}$, $v_{ji}(\phi) > 0$, may not be consistent. So the task is to synthesize from the ratios of preferences a vector that best represents the voters' preferences.

2.1. Properties of the Eigenvector Method

In general, when the voting matrix of pairwise comparison ratios is not consistent, it is not possible to infer from $a_{ij}(\phi) \equiv \frac{v_{ij}(\phi)}{v_{ji}(\phi)} > 1$ that alternative *i* defeats alternative *j*. Let $\bar{w}(\phi)$ be the principal right eigenvector of the matrix $A(\phi)$, i.e., $A(\phi)\bar{w}(\phi) = \lambda_{\max}\bar{w}(\phi)$.

<u>Definition</u>: A reciprocal pairwise voting matrix $A(\phi) = \{a_{ij}(\phi)\}$ satisfies *row dominance* if and only if for any two rows *i* and *j*, $a_{ih}(\phi) \geq a_{jh}(\phi)$ or $a_{ih}(\phi) \leq a_{jh}(\phi)$, for all *h*. Thus, a profile satisfies row dominance if its corresponding reciprocal pairwise voting matrix satisfies it.

Note that because $v_{ih}(\phi) + v_{hi}(\phi) = v_{jh}(\phi) + v_{hj}(\phi) = N$ then $a_{ih}(\phi) \geq a_{jh}(\phi)$ implies $v_{ih}(\phi) \geq v_{jh}(\phi)$. Row dominance defines a strong order on the set of alternatives.

Theorem 1: *The eigenvector method on profiles that satisfy row dominance is Condorcet.*

Definition: A voting method *f* is said to be *consistent* if and only if given two disjoint profiles ϕ' and ϕ'' on which the method yields the same consensus ordering, i.e., $f(\phi') = f(\phi'')$, it yields the same consensus ordering on the joint profile $\phi = \phi' \cup \phi''$, i.e., $f(\phi) = f(\phi') = f(\phi'')$.

Lemma 1: *The union of two disjoint profiles that satisfy row dominance and yield the same order on the alternatives satisfies row dominance and yields the same order.*

Lemma 2: *Given a profile ϕ with corresponding reciprocal pairwise voting matrix $A(\phi) = \left(\frac{v_{ij}(\phi)}{v_{ji}(\phi)}\right)$ that satisfies row dominance, if for any i and j, $a_{ih}(\phi) \geq a_{jh}(\phi)$, for all h, then $w_i(\phi) \geq w_j(\phi)$, where $\overline{w}(\phi) = \left(w_i(\phi)\right)^T$ is the principal right eigenvector of the reciprocal pairwise voting matrix $A(\phi)$.*

Theorem 2: *The eigenvector method on profiles that satisfy row dominance is consistent.*

Definition: A voting method satisfies *independence from irrelevant alternatives* (IIA) if the addition/deletion of an alternative to a profile does not alter the consensus ordering obtained from the original profile.

Let ϕ be a profile with *m* alternatives, let $A(\phi) = \left(\frac{v_{ij}(\phi)}{v_{ji}(\phi)}\right)$ be the associated reciprocal pairwise voting matrix, and let $\overline{w}(\phi)$ be the corresponding principal right eigenvector of $A(\phi)$. Let ϕ' be the profile resulting from the addition of a new alternative, let $A(\phi') = \left(\frac{v_{ij}(\phi')}{v_{ji}(\phi')}\right)$ be the associated reciprocal pairwise voting matrix, and let $\overline{w}(\phi')$ be the corresponding principal right eigenvector. The matrix $A(\phi')$ is given by $A(\phi') = \begin{pmatrix} A(\phi) & \overline{a}_{m+1}(\phi') \\ \dfrac{1}{\overline{a}_{m+1}(\phi')} & 1 \end{pmatrix}$ where $\overline{a}_{m+1}(\phi')$ is an $n \times 1$ vector given by $\overline{a}_{m+1}(\phi') = \left(\frac{v_{1,m+1}(\phi')}{v_{m+1,1}(\phi')}, \cdots, \frac{v_{n,m+1}(\phi')}{v_{m+1,n}(\phi')}\right)^T$.

Theorem 3: *The eigenvector method on profiles with row dominance satisfies independence from irrelevant alternatives.*

Corollary: *The eigenvector method on profiles that satisfy row dominance satisfies the independence of clones criterion.* ♦

A condition that is weaker than row dominance is N-dominance.

Definition: A reciprocal pairwise voting matrix $A(\phi)$ is *N-dominant* if there exists a positive integer N such that for all $n \geq N$, and any two rows *i* and *j* the nth power of $A(\phi)$, $A(\phi)^n = \left\{a_{ij}^{(n)}(\phi)\right\}$, satisfies $a_{ih}^{(n)}(\phi) \geq a_{jh}^{(n)}(\phi)$ or $a_{ih}^{(n)}(\phi) \leq a_{jh}^{(n)}(\phi)$ for all *h*.

Lemma 3: *A reciprocal pairwise voting matrix without preference loops is N-dominant.*

Lemma 4: *Let $A(\phi)$ be a reciprocal pairwise voting matrix, and let $\overline{w}(\phi) = \left(w_1(\phi), \ldots, w_n(\phi)\right)^T$ be its principal right eigenvector. Let $A(\phi)^n = \left\{a_{ij}^{(n)}(\phi)\right\}$ be the n^{th} power of $A(\phi)$. Then*

$$\lim_{n\to\infty} A(\phi)^n * Diag[A(\phi)^n]^{-1} = \overline{w}(\phi) * [\overline{w}(\phi)^T]^{-1}$$

where $Diag[A(\phi)^n] = \begin{pmatrix} a_{11}^{(k)}(\phi) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn}^{(k)}(\phi) \end{pmatrix}$.

Theorem 4: *Row dominance is a necessary and sufficient condition for the Eigenvector Method of Pairwise Voting to satisfy Independence of Irrelevant Alternatives.*

3. Pairwise Voting With Intensity Of Preferences

In elections, voting is a modified form of ranking and all the votes are equally important. However, there are situations similar to voting in which the votes are not just ordinal but each voter expresses an intensity of preference for the different candidates. For example, ranking projects for funding. The voters, in this case they are called referees, score the different projects according to some criteria. The scores are then used to rank the projects. The National Institutes of Health use such a system and so does the National Science Foundation. In general, due to the large number of projects that need to be ranked, it is possible that some referees only rank some but not all projects. In this case the projects need to be scored so that they can be compared with the scores of other similar projects and the ranks merged. But within a subgroup of projects the referees score all the projects. Thus, we can assume without loss of generality that all the referees rank all the projects.

3.1 Ranking with Complete Pairwise Comparisons

Ranking can be considered a limiting condition of pairwise comparisons. Consider two projects a_1 and a_2. Comparing them according to a criterion we can express how strongly we prefer one project to the other. For example, if a_1 is preferred to a_2 with intensity *a* the result is a reciprocal matrix of pairwise comparisons given by

$$\begin{array}{cc} & \begin{array}{cc} a_1 & a_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \end{array} & \begin{pmatrix} 1 & a \\ a^{-1} & 1 \end{pmatrix} \end{array}$$

This matrix has a principal right eigenvector given by $\begin{pmatrix} \dfrac{a}{1+a} & \dfrac{1}{1+a} \end{pmatrix}^T$ that converges to the vector $(1,0)^T$ as $a \to \infty$. Thus, the ordinal ranking of the two projects is equivalent to pairwise comparing them with an intensity of infinity of one project over another.

Given a profile $\phi = \begin{pmatrix} (n_{1>2}) & (n_{2>1}) \\ a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}$ the pairwise voting matrix is given by

$$\begin{array}{c} a_1 \\ a_2 \end{array} \begin{pmatrix} 1 & n_1(\phi)/n_2(\phi) \\ n_2(\phi)/n_1(\phi) & 1 \end{pmatrix}.$$

If all the voters that prefer a_i to a_j have an intensity of preference a_{ij} then we can represent the pairwise voting matrix as follows so that when $a_{ij} \to \infty$ we obtain the pairwise voting matrix given above:

$$\begin{pmatrix} 1 & \dfrac{\left(\dfrac{a_{1>2}}{a_{1>2}+1}\right) n_1(\phi)}{\left(\dfrac{a_{2>1}}{a_{2>1}+1}\right) n_2(\phi)} \\ \dfrac{\left(\dfrac{a_{2>1}}{a_{2>1}+1}\right) n_2(\phi)}{\left(\dfrac{a_{1>2}}{a_{1>2}+1}\right) n_1(\phi)} & 1 \end{pmatrix}$$

and the principal right eigenvector of this matrix is given by

$$\begin{pmatrix} \dfrac{\left(\dfrac{a_{12}}{a_{12}+1}\right) n_{1>2}(\phi)}{\left(\dfrac{a_{12}}{a_{12}+1}\right) n_{1>2}(\phi) + \left(\dfrac{a_{21}}{a_{21}+1}\right) n_{2>1}(\phi)} \\ \dfrac{\left(\dfrac{a_{21}}{a_{21}+1}\right) n_{2>1}(\phi)}{\left(\dfrac{a_{12}}{a_{12}+1}\right) n_{1>2}(\phi) + \left(\dfrac{a_{21}}{a_{21}+1}\right) n_{2>1}(\phi)} \end{pmatrix}$$

If all the voters have different intensity of preference $a_{i>j}^{(k)}$, then the pairwise voting matrix is given by

$$\begin{pmatrix} 1 & \dfrac{\sum_{k=1}^{n_{1>2}(\phi)}\left(\dfrac{a_{12}^{(k)}}{a_{12}^{(k)}+1}\right)}{\sum_{k=1}^{n_{2>1}(\phi)}\left(\dfrac{a_{21}^{(k)}}{a_{21}^{(k)}+1}\right)} \\ \dfrac{\sum_{k=1}^{n_{2>1}(\phi)}\left(\dfrac{a_{21}^{(k)}}{a_{21}^{(k)}+1}\right)}{\sum_{k=1}^{n_{1>2}(\phi)}\left(\dfrac{a_{12}^{(k)}}{a_{12}^{(k)}+1}\right)} & 1 \end{pmatrix}.$$

Finally, if we compare *m* projects the pairwise voting matrix with intensity of preferences would be given by

$$A(\phi)=\begin{pmatrix} 1 & \dfrac{\sum_{k=1}^{n_{1>2}(\phi)}\dfrac{a_{12}^{(k)}}{a_{12}^{(k)}+1}}{\sum_{k=1}^{n_{1<2}(\phi)}\dfrac{a_{21}^{(k)}}{a_{21}^{(k)}+1}} & \cdots & \dfrac{\sum_{k=1}^{n_{1>m}(\phi)}\dfrac{a_{1m}^{(k)}}{a_{1m}^{(k)}+1}}{\sum_{k=1}^{n_{1<m}(\phi)}\dfrac{a_{m1}^{(k)}}{a_{m1}^{(k)}+1}} \\ - & 1 & \cdots & \dfrac{\sum_{k=1}^{n_{2>m}(\phi)}\dfrac{a_{2m}^{(k)}}{a_{2m}^{(k)}+1}}{\sum_{k=1}^{n_{2<m}(\phi)}\dfrac{a_{m2}^{(k)}}{a_{m2}^{(k)}+1}} \\ \vdots & \vdots & \ddots & \vdots \\ - & - & \cdots & 1 \end{pmatrix} \tag{1.3}$$

where $n_{i>j}(\phi)$ and $a_{ij}^{(k)}$represent the number of voters that prefer *i* to *j* and the intensity with which the k^{th} voter prefers *i* to *j*, respectively. Note that when $a_{ij}^{(k)} \to \infty$, for all i and j, the pairwise voting matrix converges to the matrix

$$W(\phi)=\begin{pmatrix} 1 & w_{12}(\phi) & \cdots & w_{1m}(\phi) \\ w_{21}(\phi) & 1 & \cdots & w_{2m}(\phi) \\ \vdots & \vdots & \ddots & \vdots \\ w_{m1}(\phi) & w_{m2}(\phi) & \cdots & 1 \end{pmatrix} \tag{1.4}$$

where $w_{ij}(\phi)=\dfrac{n_{i>j}(\phi)}{n_{j>i}(\phi)},\ n_{j>i}(\phi)>0$.

3.2 Example

Consider the following voting profile

$$\begin{array}{cccc} (4) & (3) & (5) & (2) \end{array}$$
$$\phi = \begin{pmatrix} a_1 & a_2 & a_1 & a_3 \\ a_2 & a_3 & a_3 & a_1 \\ a_3 & a_1 & a_2 & a_2 \end{pmatrix}$$

The reciprocal pairwise voting matrix and its corresponding principal right eigenvector are given by

$$A(\phi) = \begin{pmatrix} 1 & 11/3 & 9/5 \\ 3/11 & 1 & 7/7 \\ 5/9 & 7/7 & 1 \end{pmatrix},\ \overline{w}(\phi) = \begin{pmatrix} 0.5605 \\ 0.1938 \\ 0.2457 \end{pmatrix}.$$

Let us assume that the voters express their intensity of preferences as given in Table 1. Synthesizing the judgments using (1.3) we get the matrix and corresponding eigenvector

$$\hat{A}(\phi) = \begin{pmatrix} 1 & 3.4547 & 1.8862 \\ 0.2895 & 1 & 0.9893 \\ 0.5302 & 1.0108 & 1 \end{pmatrix}, \hat{w}(\phi) = \begin{pmatrix} 0.5594 \\ 0.1974 \\ 0.2432 \end{pmatrix} \tag{1.5}$$

3.3. Ranking with Incomplete Pairwise Preferences

Let us now assume that the voters do not produce all possible orderings of the candidates to make it possible to have $w_{ij}(\phi) = \dfrac{n_{i>j}(\phi)}{n_{j>i}(\phi)}$, $n_{j>i}(\phi) > 0$ for all *i* and *j*. Thus, there exist some *i* and *j* such that $n_{i>j}(\phi) > 0$ but $n_{j>i}(\phi) = 0$. This can happen in three situations: (a) Nobody ranks *j* ahead of *i*; (b) *i* is not even ranked by some of the voters; and (c) some voters do not rank a candidate but there does not exist a pair *i* and *j* as in case (a). Examples of cases (a), (b) and (c) are the profiles ϕ', ϕ'' and ϕ''', respectively:

$$\begin{array}{ccc} (4) & (3) & (5) \end{array} \qquad \begin{array}{cccc} (4) & (3) & (5) & (2) \end{array} \qquad \begin{array}{cccc} (4) & (3) & (5) & (2) \end{array}$$
$$\phi' = \begin{pmatrix} a_1 & a_1 & a_3 \\ a_2 & a_3 & a_1 \\ a_3 & a_2 & a_2 \end{pmatrix} \quad \phi'' = \begin{pmatrix} a_1 & a_2 & a_1 & a_3 \\ a_2 & a_3 & a_3 & \\ a_3 & & a_2 & a_2 \end{pmatrix} \quad \phi''' = \begin{pmatrix} a_1 & a_2 & a_1 & a_3 \\ a_2 & a_3 & a_3 & a_2 \\ a_3 & & a_2 & a_1 \end{pmatrix}$$

The reciprocal pairwise voting matrices are given by

$$A(\phi') = \begin{pmatrix} 1 & 12/0 & 7/5 \\ 0/12 & 1 & 4/8 \\ 5/7 & 8/4 & 1 \end{pmatrix} A(\phi'') = \begin{pmatrix} 1 & 9/0 & 9/0 \\ 0/9 & 1 & 7/7 \\ 0/9 & 7/7 & 1 \end{pmatrix} A(\phi') = \begin{pmatrix} 1 & 9/2 & 9/2 \\ 2/9 & 1 & 7/7 \\ 2/9 & 7/7 & 1 \end{pmatrix}$$

In case (a) a way to provide a solution would be to introduce a phantom voter that prefers *j* to *i*. This will work as log as the number of voters is not too small. Case (b) can be easily handled using intensity of preferences. Case (c) does not pose problems because the pairwise voting matrix does not have any zeros. Table 2 shows the intensity of preferences of the voters for profile ϕ''. Using (1.4) the reciprocal pairwise voting matrix and corresponding eigenvector are now given by

$$\hat{A}(\phi'') = \begin{pmatrix} 1 & 2.4166 & 1.4463 \\ 0.4138 & 1 & 0.9893 \\ 0.6914 & 1.0108 & 1 \end{pmatrix}, \hat{w}(\phi'') = \begin{pmatrix} 0.4821 \\ 0.2359 \\ 0.2819 \end{pmatrix}.$$

3.4. Ranking from Weights

Profiles do not always have to be of an ordinal type. Assume that each voter has a set of priorities assigned to the alternatives. Let $\overline{w}^{(k)}(\phi)$ be the priorities of the k^{th} voter in profile ϕ. From these priorities we can estimate the reciprocal pairwise matrix of preferences of the voters $a_{ij}^{(k)}(\phi)$ by $\hat{a}_{ij}^{(k)}(\phi) = \dfrac{w_i^{(k)}(\phi)}{w_j^{(k)}(\phi)}$. The reciprocal pairwise voting matrix is obtained using (1.3). For profile ϕ, the reciprocal pairwise voting matrix and corresponding eigenvector are given by

$$\hat{A}(\phi) = \begin{pmatrix} 1 & 3.0950 & 1.7722 \\ 0.3231 & 1 & 0.9509 \\ 0.5643 & 1.0516 & 1 \end{pmatrix}, \hat{w}(\phi) = \begin{pmatrix} 0.5379 \\ 0.2058 \\ 0.2563 \end{pmatrix}.$$

4. Conclusions

We have developed a method of obtaining the winner of an election by using reciprocal pairwise comparisons. The literature has used pairwise comparisons but they have always been of an additive nature. We have shown that because the resulting matrix of paired comparisons is a positive reciprocal matrix, the winner of the election is given by the principal right eigenvector of the matrix. This method has some desirable properties when the reciprocal pairwise voting matrix satisfies the condition of row dominance, a necessary and sufficient condition for satisfying the axiom of independence from irrelevant alternatives. We can now show that voting profiles satisfying the row dominance condition do not yield rank reversals when a candidate drops out of the race.

Table 1. Profile ϕ voters' preferences.

Orderings	Voter	Pairwise Preference Matrices		
	1	1	3	5
1 2 3		0.33333333	1	5
		0.2	0.2	1
	2	1	5	7
		0.2	1	5
		0.14285714	0.2	1
	3	1	2	5
		0.5	1	5
		0.2	0.2	1
	4	1	4	8
		0.25	1	5
		0.125	0.2	1
2 3 1	5	1	0.0001	0.0001
		10000	1	3
		10000	0.33333333	1
	6	1	0.0001	0.0001
		10000	1	2
		10000	0.5	1
	7	1	0.0001	0.0001
		10000	1	2
		10000	0.5	1
1 3 2	8	1	7	3
		0.14285714	1	0.2
		0.33333333	5	1
	9	1	5	3
		0.2	1	0.2
		0.33333333	4	1
	10	1	6	3
		0.16666667	1	0.2
		0.33333333	3	1
	11	1	4	3
		0.25	1	0.2
		0.33333333	3	1
	12	1	5	4
		0.2	1	0.2
		0.25	2	1
3 1 2	13	1	0.0001	0.0001
		10000	1	0.14285714
		10000	7	1
	14	1	0.0001	0.0001
		10000	1	0.14285714
		10000	4	1

Table 2. Profile ϕ'' voters' preferences with incomplete rankings

Orderings	Voter	Pairwise Preference Matrices		
	1	1	3	5
1 2 3		0.33333333	1	5
		0.2	0.2	1
	2	1	3	5
		0.33333333	1	5
		0.2	0.2	1
	3	1	3	5
		0.33333333	1	5
		0.2	0.2	1
	4	1	3	5
		0.33333333	1	5
		0.2	0.2	1
2 3 1	5	1	0.14285714	0.2
		7	1	5
		5	0.2	1
	6	1	0.14285714	0.2
		7	1	5
		5	0.2	1
	7	1	0.14285714	0.2
		7	1	5
		5	0.2	1
1 3 2	8	1	5	3
		0.2	1	0.2
		0.33333333	5	1
	9	1	5	3
		0.2	1	0.2
		0.33333333	5	1
	10	1	5	3
		0.2	1	0.2
		0.33333333	5	1
	11	1	5	3
		0.2	1	0.2
		0.33333333	5	1
	12	1	5	3
		0.2	1	0.2
		0.33333333	5	1
3 1 2	13	1	3	0.2
		0.33333333	1	0.14285714
		5	7	1
	14	1	3	0.2
		0.33333333	1	0.14285714
		5	7	1

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