ON THE SAMPLING DISTRIBUTION OF THE RANDOM CONSISTENCY INDEX OF ANALYTIC HIERARCHY PROCESS (AHP)

V.M. Rao Tummala†
Department of Manufacturing Engineering
City Polytechnic of Hong Kong, Hong Kong

and

Yat-wah Wan
Department of Industrial Engineering
Hong Kong University of Science and Technology, Hong Kong

† On Leave form the College of Business, Eastern Michigan University, Ypsilanti, MI, USA

Abstract

The mean random consistency index (MRCI) problem is an important problem that is addressed by several AHP researchers. The MRCI is the basis for defining the consistency ratio, CR(n) which measures the consistency of the judges or the decision makers who assign relative scales in a pair wise fashion. It is defined as the average value of the consistency indices, CI(n) which are generated at random by N pairwise comparison matrices of size n. In this paper, we first find a closed-form expression for the largest eigenvalue of a three dimensional random pairwise comparison matrix. Then by simulation analysis, we obtain the sampling distribution and the corresponding mean and the standard deviation of the distribution of the consistency index. The mean and the standard deviations are compared with those obtained by others.

Key Words: Analytical Hierarchy Process, Largest Eigenvalue Problem, Consistency Index, Mean Random Consistency Index, and Consistency Ratio.

1. Introduction

The mean random consistency index (MRCI) problem is an important problem that is addressed by several AHP researchers [Saaty (1988), Golden and Wang (1990), Lane and Verdini (1989), Vargas (1982), and Tummala and Wan (1994)]. The MRCI is the basis for defining the consistency ratio, CR(n) which measures the consistency of the judges or the decision makers who assign relative scales in a pair wise fashion (forming a pairwise comparison matrix (PCM) of size n) with respect to n attributes, A_1, A_2, ..., A_n, of one level given the attributes of the next higher level of the hierarchy associated with a decision problem. The main purpose of AHP is to determine the (local) priority vector $w = (w_1, w_2, ..., w_n)$ of normalized weights corresponding to A_1, A_2, ..., A_n of each level and hence the global priority weights associated with the alternatives occupying the last level

of the hierarchy. These global priority weights will then be used to rank the alternatives.

If the judges assign relative scales by comparing each pair A_i and A_j of the attributes as to the degree by which A_i dominates A_j as $a_{ij} = w_i/w_j$, and form the pairwise comparison matrix A, then w can be found by solving the corresponding largest eigenvalue problem as

$$Aw = nw$$

where $A=((a_{ij}))$. The relative scales are assigned by following the nine-point scale taking values in Ω = {1,2, ... 9,1,1/2,1/3, ... 1/9} [Saaty (1988)]. Notice that, by definition, $a_{ij} = 1/a_{ji}$, $a_{ii} = 1$, and $a_{ij} > 0$ for i and j in Ω. Thus the PCM A is a perfectly consistent matrix. Therefore, the largest eigenvalue of A, namely, λ_{max} should be equal to n; that is, $\lambda_{max}=n$. Thus solving

$$Aw = nw$$

to find w is equivalent to solving

$$Aw = \lambda_{max} w$$

The consistency index to measure the consistency of the deviation of λ_{max} from n is given by

$$CI(n) = (\lambda_{max}-n)/(n-1)$$

Based on this measure, Saaty has defined the consistency ratio for any PCM as

$$CR(n) = CI(n)/MRCI(n)$$

and recommended that $CR(n) \leq 0.10$ for an evaluator to be consistent in assigning relative scales [Saaty (1988)]. The MRCI(n) in the above equation is called the mean random consistency index which is defined as the average value of the consistency indices CI(n) which are generated by randomly selecting N pairwise comparison matrices of size n; that is,

$$MRCI(n) = (\bar{\lambda}_{max}-n)/(n-1) \qquad (1)$$

where $\bar{\lambda}_{max}$ is the average or expected value of λ_{max}.

Although sets of MRCI(n)s have been widely reported in the literature, there have been no replicable simulations described [Noble (1990), Sanchez and Noble (1991)]. Our objective in this paper is to develop an algorithm based on "Power Method" to improve the computational accuracy of MRCI(n) [Golub and Van Loan (1989)]. Some authors have used the Power Method but did not

design the experiment to run simulations in terms of the random seed as well as the number of different perturbations of the elements of A, to obtain the sampling distribution of CI(n) [Noble (1990), Sanchez and Noble (1991)]. We conducted simulation experiments in this paper not only to find the sampling distribution of CI(n) but also to conduct relevant statistical analysis. We also have provided the comparative analysis with the results obtained by others in this area.

2. Present Studies on MRCI(n)

From Eq.(1) we see that the consistency problem of AHP really depends on the computation of MRCI(n). As mentioned above, several authors have addressed this problem and obtained results as presented in Table 1 [Saaty (1988), Golden and Wang (1990), Lane and Verdini (1989), Noble (1990)]. Saaty (at Wharton) and Uppuluri (at Oak Ridge) were among the first to conduct simulation experiments with 500 and 100 runs, respectively [Saaty (1988)]. Lane and Verdini, Golden and Wang, and Noble later followed with 2500, 1000, and 5000 simulation runs, respectively. Golden and Wang, and Lane and Verdini did not explain how they have generated the random PCM matrices and found the corresponding MRCI(n)s. Presumably, they followed the procedure which was used by Saaty and Uppuluri.

Table 1. MRCI(n) and SD(CI(n)) from Previous Studies

	MRCI(n)					SD(CI(n))			
Source	Golden & Wang	Lane & Verdini	Noble	Wharton	Oak Ridge	Golden & Wang	Lane & Verdini	Noble	Oak Ridge
# of Run	1000	2500	5000	500	100	1000	2500	5000	100
n									
3	0.5799	0.52	0.49	0.58	0.382	0.7381	0.70	0.67	0.5165
4	0.8921	0.87	0.82	0.90	0.946	0.6299	0.63	0.60	0.6580
5	1.1159	1.10	1.03	1.12	1.220	0.5243	0.51	0.49	0.5280
6	1.2358	1.25	1.16	1.24	1.032	0.4120	0.40	0.40	0.4247
7	1.3322	1.34	1.25	1.32	1.468	0.3358	0.33	0.33	0.3478
8	1.3952	1.40	1.31	1.41	1.402	0.2895	0.28	0.28	0.2719
9	1.4537	1.45	1.36	1.45	1.350	0.2382	0.24	0.24	0.2190
10	1.4882	1.49	1.39	1.49	1.464	0.2131	0.21	0.21	0.1691
11	1.5117	N.A	1.42	1.51	1.576	0.1946	N.A	0.19	0.2161
12	1.5356	1.54	1.44		1.476	0.1663	0.17	0.17	0.5634
13	1.5571	N.A	1.46		1.564	0.1540	N.A	0.15	0.1750
14	1.5714	1.57	1.48		1.568	0.1360	0.14	0.14	0.1483
15	1.5831	N.A	1.49		1.586	0.1311	N.A	0.13	0.1457

Saaty, on the other hand, found the λ_{max} and the associated normalized eigenvector w by raising pairwise comparison matrices to increasing powers and normalizing the resulting system as

$$w = \lim_{k \to \infty} \frac{A^k e}{e^T A e}$$

where A is the pairwise comparison matrix and e is the vector

consisting of 1's in each entry. He used these λ_{max}s to determine MRCI(n). Unfortunately this method is relatively unstable numerically. While noting this instability, Noble used the Power Method to find λ_{max} and conducted simulation experiments to determine the corresponding MRCI(n)s. However, as mentioned above, she did not design simulation experiment in terms of random seeds and did not statistically analyze the results [Noble (1990)].

The Power Method and Saaty's procedure are similar; both depend upon the ratio between the λ_{max} and the next largest eigenvalue and both have an $O(1/\lambda^2_{max})$ convergence rate. However, the Power Method is numerically more stable. Basically, it normalizes trail vectors for the dominating eigenvector at every iteration, while Saaty's approach normalizes only at the very last step. Therefore, we use the Power Method in finding λ_{max} and conduct simulation experiments to analyze the results by a rigorous statistical analysis. Before we do this, we shall find the closed-form solution for obtaining λ_{max} for a three-dimensional random PCM. We can use this expression to find the expected value and the standard deviation of λ_{max}.

3. Exact Solution of λ_{max} for a 3-Dimensional Random PCM

Lane and Verdini found the exact value of $\bar{\lambda}_{max}$ by solving all the 4913 characteristic equations of a 3-dimensional PCM [Lane and Verdini (1989). Due to the special structure of the PCM, we can find a closed form expression for λ_{max} as follows. Without the loss of generality assume that

$$A = \begin{pmatrix} 1 & X & 1/Z \\ 1/X & 1 & Y \\ Z & 1/Y & 1 \end{pmatrix}$$

where X, Y, Z are identically and independently distributed random variables with Uniform distribution over Ω. The characteristic equation corresponding to A is given by

$$(1-\lambda)^3 + XYZ + 1/(XYZ) - 3(1-\lambda) = 0$$

Solving this cubic equation, according to a standard procedure [Tuma (1987)]

$$\lambda_{max} = 1 + (XYZ)^{1/3} + 1/(XYZ)^{1/3} \qquad (2)$$

This expression for λ_{max} is not new. Vargas obtained the same result for a 3-dimensional PCM by decomposition [Vargas (1982)]. Since he worked with deterministic matrices, he did not carry out the mean and variance analysis as we will show below.

By definition, $\bar{\lambda}_{max} = E(\lambda_{max})$. Using Eq. (2) and simplifying, we obtain the expected value as

$$\bar{\lambda}_{max,3} = 1+2E^3(X^{1/3}) \qquad (3)$$

The equality in the above equation holds true because the random variables are identically and independently distributed and have a distribution which is symmetric with respect to the reciprocal operation. In a similar fashion, we can find the variance, $Var(\lambda_{max})$ as

$$Var(\lambda_{max}) = 2\left[Var^3(X^{1/3}) + 3E^3(X^{1/3})Var^2(X^{1/3}) + 3E^4(X^{1/3})Var(X^{1/3}) + 1 - E^6(X^{1/3})\right] \qquad (4)$$

Since the distribution of X is Uniform, we see that $E(X^{1/3})=1.143$ and $Var(X^{1/3})=0.323$. Therefore, we find that $\bar{\lambda}_{max,3}=3.983$ and $Var(\bar{\lambda}_{max,3})=1.735$. Substituting these values in Eqs.(3) and (4), we obtain MRCI(3)=0.491, Var(CI(3))=0.434 and SD(CI(3)=0.659.

It is interesting to note only Noble has obtained this exact value for MRCI(3). Lane and Verdini is about 6% over the exact value. The rest are even worse. Both Golden and Wang, and Saaty's are over by 18% and the Uppuluri's by 22%. The same is true for variance even though the values are different.

4. Sampling Distribution of CI(n) By Simulation Analysis

We generate the (pseudo-) random numbers $\{x_i\}$ based on the congruential generator

$$x_i = (16807 \cdot x_{i-1}) \bmod 2147483647$$

with the (pseudo-) zero-one uniform variates $\{u_i\}$ given by

$$u_i = x_i/2147483647$$

This generator is a popular one in the sense that it has full range, and has acceptable statistical performance [Noble (1990), Ripley (1987)].

For each n, $3 \le n \le 15$, we shall use this generator to generate N_n of n-dimensional pairwise comparison matrices at random. The λ_{max} of each PCM will be found through the Power Method. The sampling distribution of CI(n) will then be found from using all N_n of λ_{max}s.

The number N_n is chosen to make both the results have small absolute and relative errors. For absolute error, we require that

the width of the (approximate) confidence interval of $\bar{\lambda}_{max}$ is no greater than 0.01. Let S_n be the sample standard deviation of λ_{max} for an n-dimensional PCM. Then it is known that

$$\frac{MRCI(n) - \bar{\lambda}_{max}}{S_n/\sqrt{n}}$$

converges in distribution to standard normal. Thus, for large n, the width of the 100(1-α) percent confidence interval for MRCI(n) is approximately given by

$$\frac{2\, z_{\alpha/2} s_n}{\sqrt{n}}$$

where $z_{\alpha/2}$ is the 100(1-α/2) percentile of standard normal distribution, and s_n is the realization of S_n. Therefore, we obtain

$$N_n \geq 40000 z_{\alpha/2}^2 s_n^2 \qquad (5)$$

to insure that the width of the confidence interval is less than 0.01, where s_n is taken to be the maximum value of the SD(CI(n))s obtained by Golden and Wang, Lane and Verdini, and Noble. Table 2 gives the minimal integral N_n that satisfies Eq. (5) for $z_{\alpha/2}$=2.575 (α=0.01). Note that this implies N_n is decreasing with n, which is different form all the previous studies mentioned earlier.

For relative error, we want to ensure that the statistical error does not affect the critical value too much, so that, we can take a width which is less than β (0<β<1) of the critical value, δMRCI(n). Thus

$$N_n \geq \frac{5.15 s_n^2}{\delta^2 \beta^2 MRCI(n)^2} \qquad (6)$$

The third row of Table 2 gives the corresponding values of N_n with δ=0.1, β=0.05, and MRCI(n) being the minimum of the three values obtained by Golden and Wang, Lane and Verdini, and Noble. We now use Eqs.(5) and (6), to find N_n from each equation and determine the maximum of the two values. This value is taken to be the value for our N_n (see row four of Table 2). Note that our values of N_n for n≤14 are substantially greater than those used in the previous studies.

We need n(n-1)/2 random numbers to generate one random n-dimensional PCM. With the N_n specified in Table 2, the total

number of random numbers used is around 8,500,000, which is substantially below the cycle of this generator. Thus, we can use distinct random numbers through our simulation which can reduce the possible dependence of the repetition of random numbers in the computation of λ_{max}.

Table 2. Values of N_n

n	3	4	5	6	7	8	9	10	11	12	13	14	15
Nn (Eq.(11))	144,493	105,268	72,908	45,021	29,908	22,229	15,277	12,044	10,043	7,666	6,291	5,199	4,559
Nn (Eq.(12))	467,419	121,597	53,376	25,987	14,867	10,061	6,416	4,842	3,869	2,872	2,292	1,843	1,595
Nn used	470,000	122,000	73,000	46,000	30,000	22,500	15,500	12,500	10,500	7,700	6,300	5,200	4,600
# of random Numbers used	1,410,000	732,000	730,000	690,000	630,000	630,000	558,000	562,500	577,500	508,200	491,400	473,200	483,000

We use N_n of Table 2 and simulate the corresponding sampling distributions of CI(n) for n=3,4,5,...,15. The percentile distributions are shown in Table 3 while the corresponding cumulative distributions of CI(n) are described in Fig. 1. Table 4 describes the values for mean and the standard deviation of the random consistency index for n=3,4,5,...,15.

Table 3. Percentiles of Random Consistency Indices

	percentile												
n	1	5	10	20	30	40	50	60	70	80	90	95	99
3	0.000206	0.001295	0.003385	0.036330	0.081417	0.139980	0.217365	0.308589	0.500186	0.855528	1.313824	2.082668	2.890111
4	0.036588	0.100360	0.161080	0.270895	0.393772	0.533655	0.696294	0.895627	1.127098	1.383560	1.742273	2.025140	2.392765
5	0.160375	0.288944	0.394645	0.566346	0.724208	0.871419	1.020407	1.171382	1.326429	1.499720	1.721690	1.901647	2.188000
6	0.332000	0.501611	0.642719	0.818455	0.953385	1.068227	1.174000	1.292432	1.403690	1.529846	1.703403	1.834444	2.100000
7	0.502750	0.711750	0.837875	0.985706	1.096267	1.183842	1.268381	1.356458	1.450000	1.545552	1.676800	1.795000	2.001875
8	0.659667	0.857250	0.963125	1.089480	1.180625	1.258333	1.332519	1.404639	1.484053	1.565786	1.678500	1.767615	1.947600
9	0.789200	0.960250	1.057750	1.166765	1.242500	1.312450	1.375750	1.436095	1.502130	1.574040	1.673154	1.753250	1.896000
10	0.908750	1.058500	1.131727	1.224900	1.299130	1.356875	1.411788	1.462893	1.516727	1.582895	1.671267	1.743167	1.877333
11	0.990500	1.127400	1.191000	1.274846	1.335880	1.388313	1.435464	1.482370	1.531278	1.589389	1.670154	1.737500	1.869000
12	1.052000	1.181000	1.238250	1.310467	1.364700	1.413407	1.456278	1.502267	1.548095	1.598625	1.672125	1.735571	1.826750
13	1.120000	1.217538	1.272727	1.343286	1.389750	1.432286	1.472000	1.514188	1.556895	1.606300	1.672167	1.725500	1.812667
14	1.171000	1.261000	1.309600	1.371000	1.412800	1.451667	1.488667	1.527867	1.566000	1.609600	1.672400	1.722667	1.803500
15	1.194500	1.291000	1.336000	1.392917	1.430846	1.465636	1.500050	1.537750	1.568615	1.611667	1.668667	1.712000	1.796000

5. Conclusions

As in the case of 3-dimensional random PCM, our results are almost same of those of Noble. However, neither Noble nor Sanchez and Noble determined the sampling distribution of CI(n). The MRCI(n) values are smaller than those obtained by Golden and Wang, Lane and Verdini, Saaty, and Uppuluri for n=3,4,5,...,15. The same is true for SD(CI(n)). This indicates that the Power Method is more accurate in computing λ_{max}s and hence MRCI(n). The smaller values of MRCI(n) means higher values for CR(n) which necessitates the revision of using Saaty's rule of CR(n) ≤ 0.10 in assessing the consistency of decision makers. Also, our analysis supports the assertion made by Noble that the use of a numerically unstable algorithm will bring about rounding errors that will overstate the value of the mean, whereas the use of a more numerically stable

algorithm would result in values closer to the true mean for randomly generated pairwise comparison matrices.

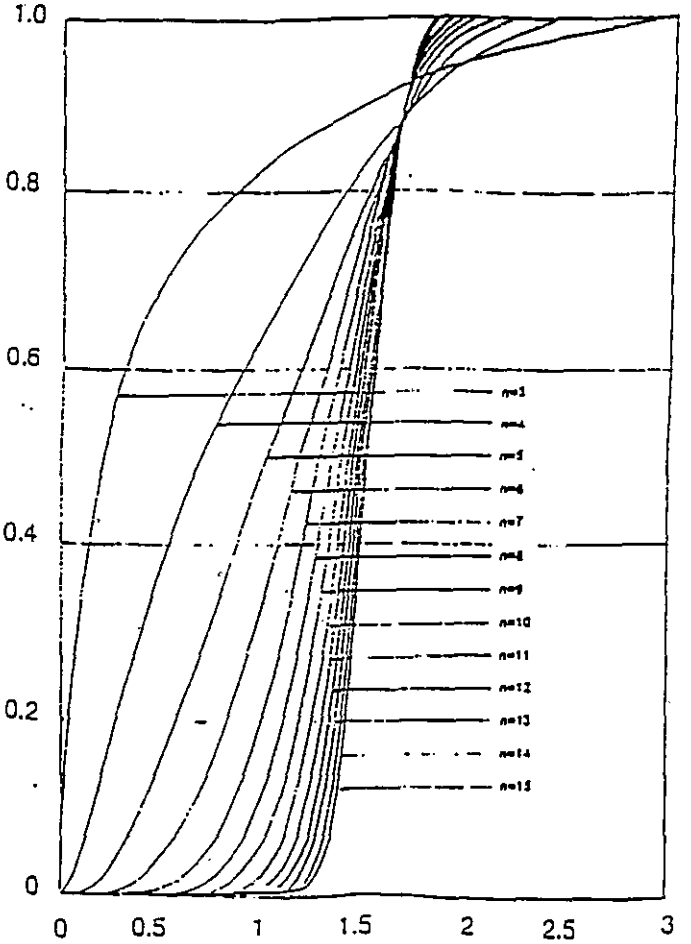

Figure 1. Cumulative Probability Distributions of CI(n). (n=2,4,5...15)

Table 4. Values of MRCI(n) and SD(CI(n))

n	N_n	$\overline{\lambda_n}$	SD(λ_{max})	MRI(n)	SD(CI(n))
3	470,000	4.000	1.348	0.500	0.674
4	122,000	6.502	1.816	0.834	0.605
5	73,000	9.183	1.996	1.046	0.499
6	46,000	11.891	2.015	1.178	0.403
7	30,000	14.601	1.970	1.267	0.328
8	22,500	17.281	1.947	1.326	0.278
9	15,500	19.956	1.913	1.369	0.239
10	12,500	22.650	1.886	1.406	0.210
11	10,500	25.335	1.873	1.433	0.187
12	7,700	28.019	1.850	1.456	0.168
13	6,300	30.684	1.851	1.474	0.154
14	5,200	33.380	1.799	1.491	0.138
15	4,600	36.012	1.810	1.501	0.129

References

[1] Golden, B.L., and Wang, Q., "An Alternative Measure of Consistency", in:B.L. Golden, E.A. Wasil and P.T. Harker (eds.), *Analytic Hierarchy Process: Applications and Studies*, Springer-Verlag: New York 1990, 68-81.

[2] Golub, G.H., and Van Loan, C.F., *Matrix Computations*, John Hopkins University Press, Baltimore, 1989.

[3] Lane, E.F., and Verdini, W.A., "A Consistency Test for AHP Decision Makers", *Decision Sciences* 20 (1989) 575-590.

[4] Noble, E.E., *Consistency in the Analytic Hierarchy Process*, Masters Thesis, Department of Operations Research and Information Systems, Eastern Michigan University, Ypsilanti, Michigan, USA, 1990.

[5] Ripley, B.D., *Stochastic Simulation*, John Wiley, New York, 1987.

[6] Saaty, T.L., *Multicriterion Decision Making: The Analytic Hierarchy Process - Planning, Priority Setting, Research Allocation*, RWS Publishers, Pittsburgh, PA, USA, 1988.

[7] Sanchez, P.P., and Noble, E.E.,"On the Information Content of a Consistent Pairwise Comparison Matrices of an AHP Decision Maker", *Proceedings of the 23rd Annual Meeting of the Decision Sciences Institute*, (1991). [8] Tummala, V.M.R., and Wan, Y.W., "Analytic Hierarchy Process (AHP) in Practice : A Survey of Applications and Recent Developments", *Mathematical Modelling and Scientific Computing* (to appear), 4 (1994).

[9] Tuma, J.J., *Engineering Mathematics Handbook*, 3rd Ed. McGraw-Hill, New York, 1987.

[10 Vargas, L.G., "Reciprocal Matrices With Random Coefficients", *Mathematical Modelling*, 3 (1982) 69-81.