SENSITIVITY ANALYSIS USING PRIORITY REACHABILITY MATRIX FOR THE ANALYTIC HIERARCHY PROCESS

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ABSTRACT

This paper treats of sensitivity analysis relating to priority which is playing an important role in the AHP. Namely, it examines theoretically, when the priority values of criteria of a certain level in hierarchy change, how these changes will affect the priority values of criteria or alternatives of other levels as well as to what degree they will affect the said priority values, and further, whether it is capable of causing rank reversal among the alternatives. Consequently, by deriving some theorems, this paper clarifies that all these problems can easily be solved analytically, using the priority reachability matrix newly defined in this paper. Further, by applying these theorems actually to a certain dwelling selection problem, this paper verifies its usefulness.

1. INTRODUCTION

The Analytic Hierarchy Process (AHP)(Saaty 1980) developed by T.L.Saaty as a support method for multiobjective decision making has recently been attracting public attention in many scientific fields such as system engineering, operations research, management science, and so on (Saaty 1983, Manabe 1986). The AHP is simple in its procedure, easily comprehensible even by beginners, and further, easily applicable to unquantifiable decision problems, compared with hitherto-existing various methods for multiobjective decision problems. For this reason, numerous applications of the AHP have been reported to date (Tone 1986 etc.).

The process of the AHP can be classified in summary into the following three steps:
[Step 1] The decision problem is expressed in the form of a hierarchy in such a way that an overall goal exists in the first level, the criteria and the subcriteria exist in the second level and thereafter, and the alternatives exist in the lowest level. Namely, the hierarchy is composed of levels whose detail increases from top to bottom.

[Step 2] The degree of importance of each element of the k-th level, $k=2,\dots,L$ with respect to the j-th element s_{k-1j}, $j=1,\dots,n$ of the (k-1)th level (one at a time) is questioned to the decision maker(s) by pairwise comparison. Then, the results thus obtained are summarized in the form of a pairwise comparison matrix (reciprocal matrix). If and only if this matrix is consistent, then eigenvector to maximum eigenvalue of the matrix is considered as the direct priority vector $a_{kj}=(a_{1j},\dots,a_{ij},\dots,a_{n_k j})^T$ of elements of the k-th level with respect to the s_{k-1j}, where the vector a_{kj} is normalized as follows:

$$\sum_{i\in I_{kj}} a_{ij} = 1 \quad \text{and} \quad a_{ij} \geq 0\,, \quad k=2,\dots,L,\ j=1,\dots,n_{k-1}, \tag{1}$$

where L is the number of levels in a hierarchy, n_{k-1} is the number of elements of the (k-1)th level, and I_{kj} is the index set of elements s_{ki} connected directly with the j-th element s_{k-1j} of the (k-1)th level. $I_{kj} \subset \{1, 2, \ldots, n_k\}$.

[Step 3] From these a_{kj}, $j=1,\ldots,n_{k-1}$, the direct priority matrix $A_k=[a_{k1}, \ldots, a_{kj}, \ldots, a_{kn_{k-1}}]$ of the k-th level is made. Then, using these matrices A_k, $k=2,\ldots,L$, the composite priority vector $w_k=(w_{k1},\ldots,w_{kn_k})^T$ of each level with respect to the overall goal is composed by the following principle:

$$w_k = A_k w_{k-1} = A_k A_{k-1} \cdots A_2 w_1, \quad k=2,\ldots,L. \tag{2}$$

Because the first level usually has a single overall goal, w_1 is scalar and its priority value is assumed to equal unity, i.e., $w_1 = 1$. Therefore, from (2), the composite priority vector w_L of the alternatives becomes as follows:

$$w_L = A_L A_{L-1} \cdots A_2. \tag{3}$$

When we use the AHP actually, in Step 2, we have often difficulty in deriving the consistent priority values from the decision maker. Namely, the decision maker must, on his subjective judgement, answer the degree of relative importance of each element with numerical values from 1 to 9 and their reciprocals, but it often happens that he can have no confidence in answering it. In such a case, if he can easily make so-called sensitivity analysis of priority, such as (i) how the result of the judgement by pairwise comparison will affect the priority values, and also (ii) how the change of these prioritiy values will affect the composite priority values of the elements of other levels, it is very useful. As regards sensitivity analysis on (i), the method (Vargas 1983) of using the Hadamard product and the methods (Harker 1985) of using the partial derivative of pairwise comparison matrix have already been studied, but as for sensitivity analysis on (ii), it has not been throughly researched yet.

At such a background, this paper treats of sensitivity analysis relating to (ii). Namely, in this paper we examine theoretically, when the priority values of criteria of a certain level in the hierarchy change, how these changes will affect the priority values of criteria or alternatives of other levels as well as to what degree they will affect the said priority values, and further, whether it is capable of causing rank reversal among the alternatives. Consequently, by deriving some theorems, we clarifies that all these problems can easily be solved analytically, using the priority reachability matrix newly defined in this paper. Further, by applying these theorems actually to a certain dwelling selection problem, we verifies its usefulness.

2. PRIORITY REACHABILITY MATRIX AND ITS PROPERTIES

Definition 1: Using priority matrices A_k, $k=2,\ldots,L$, we make a $\sum_{k=1}^{L} n_k \times \sum_{k=1}^{L} n_k$ block matrix A^* as follows:

$$
A = \begin{bmatrix} 0 & \cdots\cdots & & & 0 \\ A_2 & 0 & & & \vdots \\ 0 & A_3 & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_L & 0 \end{bmatrix} \begin{matrix} \}n_1 \\ \}n_2 \\ \}n_3 \\ \vdots \\ \}n_L \end{matrix} \quad (4)
$$

$$
\underbrace{\ }_{n_1}\ \underbrace{\ }_{n_2}\cdots\underbrace{\ }_{n_{L-1}}\ \underbrace{\ }_{n_L}
$$

Then we define a $\sum_{k=1}^{L} n_k \times \sum_{k=1}^{L} n_k$ block matrix $M=[M_{pq}]$, $p,q=1,\ldots,L$, obtained by the following equation as priority reachability matrix.

$$
M = I + A + A^2 + \ldots + A^{L-1}, \quad (5)
$$

where I is an identity matrix.

This priority reachability matrix has the following properties.

Property 1: The priority reachability matrix M is a block lower triangular matrix. Namely, each $n_p \times n_q$ block M_{pq}, $p,q=1, \ldots,L$, is as follows:

$$
M_{pq} = 0 \qquad \text{if } p < q, \quad (6)
$$

$$
M_{pq} = I \qquad \text{if } p = q, \quad (7)
$$

$$
M_{pq} = A_p A_{p-1} \cdots A_{q+1} \qquad \text{if } p > q. \quad (8)
$$

Especially, from (2), block M_{p1}, $p=2,\ldots,L$, is

$$
M_{p1} = A_p A_{p-1} \cdots A_2 = w_p. \quad (9)
$$

(Proof is omitted)

Property 2: Every column sum of each block M_{pq} ($p \geq q$) is 1.

(Proof is omitted)

(8) and (9) are important especially. (8) means that block M_{pq} ($p>q$) indicates composite priorities of the elements (i.e., subcriteria or alternatives) of the p-th level with respect to the elements (i.e., criteria or subcriteria) of the q-th level. Also, (9) means that block M_{p1} indicates composite priorities of the elements of the p-th level with respect to the overall goal. Therefore, composite priorities w_L of the alternatives for the overall goal appear in the block M_{L1}. Namely, by calculating the priority reachability matrix, it is possible to examine the order of the subcriteria or the alternatives for each criterion, so that it is possible to make further scrutiny.

3. SENSITIVITY ANALYSIS OF PRIORITY

3.1 Sensitivity analysis on local change of a_{kj}

In this section we consider the case where the priority vector a_{kj} of the elements of the k-th level with respect to the j-th element s_{k-1j} of the (k-1)th level changes from the present value a_{kj}^* to a certain value $a_{kj}^* + \Delta a_{kj}$. In this case, changes of the priority reachability matrix M and the composite priority vector w_L of the alternatives can be summarized into the following two theorems, respectively.

Theorem 1: When the value of a_{kj} changes from its present value a^*_{kj} to $a^*_{kj} + \Delta a_{kj}$, only the blocks M_{pq} which satisfies $p \geq k$ and $q \leq k-1$ change from the present value M^*_{pq} to

$$M^*_{pq} + \Delta M_{pq} = M^*_{pq} + M^*_{pk} \Delta a_{kj} [M^*_{k-1q}]_j , \tag{10}$$

where $[M^*_{k-1q}]_j$ is a $1 \times n_q$ vector consisiting of the j-th row of the matrix M^*_{k-1q}.
(Proof is omitted)

On the other hand, change of composite priority vector w_L of the alternatives becomes as follows, as a special case in Theorem 1.

Theorem 2: When the value of a_{kj} changes from its present value a^*_{kj} to $a^*_{kj} + \Delta a_{kj}$, the composite priority vector w_L of the alternatives with respect to the overall goal changes from the present value w^*_L to

$$w^*_L + \Delta w_L = w^*_L + w^*_{k-1j} M^*_{Lk} \Delta a_{kj} , \tag{11}$$

where w^*_{k-1j} is the present composite priority value of the element s_{k-1j}.
(Proof is omitted)

3.2 Sensitivity analysis on global change of a_{kj}

In 3.1, we considered the case where the value of a_{kj} changes from the present value a^*_{kj} to one specific value $a^*_{kj} + \Delta a_{kj}$. In this section, we expand it further, and consider in what region the composite priority vector w_L of the alternatives will change when a_{kj} changes throughout its whole domain.

Theorem 3: When a_{kj} changes throughout the whole domain defined by (1), namely,

$$\{ a_{kj} \mid \sum_{i \in I_{kj}} a_{ij} = 1 \text{ and } a_{ij} \geq 0 \}, \tag{12}$$

the composite priority vector w_L of the alternatives changes in the following region:

$$\Gamma_L(a_{kj}) = \{ w_L \mid w_L = \sum_{i \in I_{kj}} b_i a_{ij}, \ \sum_{i \in I_{kj}} a_{ij} = 1, \ a_{ij} \geq 0 \} , \tag{13}$$

where the $n_k \times 1$ vector b_i is

$$b_i = M^*_{Lk} [A^*_k]_i w^*_{k-1} , \tag{14}$$

and the matrix $[A^*_k]_i$ is a $n_k \times n_{k-1}$ matrix in which only the j-th column of the matrix A^*_k is replaced by $n_k \times 1$ fundamental vector e_i having 1 in the i-th entry and 0 in the other entries, namely,

$$[A^*_k]_i = [a^*_{k1} \cdots a^*_{kj-1} \ e_i \ a^*_{kj+1} \cdots a^*_{kn_{k-1}}] . \tag{15}$$

(Proof is omitted)

Theorem 3 means that the domain of a_{kj} is mapped on the convex region $\Gamma_L(a_{kj})$ formed by extreme points b_i, $i \in I_{kj}$ in the space of composite priorities of the alternatives. Namely, the larger this mapping region

$\Gamma_L(a_{kj})$ is, the more largely w_L changes according to slight change of a_{kj}. Therefore, in this paper, we define sensitivity coefficient of w_L to a_{kj} as follows:

Definition 2: $\alpha(a_{kj})$ defined by the following equation is called the sensitivity coefficient of w_L to a_{kj}.

$$\alpha(a_{kj}) = \sqrt{\frac{1}{n(I_{kj})-1} \sum_{i \varepsilon I_{kj}} (b_i - \bar{b})^T(b_i - \bar{b})}, \tag{16}$$

where $n(I_{kj})$ is the number of members in the set I_{kj}. Also, b_i is the vector given by (14), and $\bar{b}$ is the mean of b_i's numbering $n(I_{kj})$.

Above, we considered sensitivity of w_L to global change of a_{kj}. However, simply because the value of sensitivity coefficient of w_L to a certain a_{kj} is large, rank reversal among the alternatives to the change of this a_{kj} does not necessarily occur. Lastly, therefore, we summarize in the following theorem, in what cases rank reversal among the alternatives does not occur.

Theorem 4: In the vectors b_i, $i \varepsilon I_{kj}$ given by (14), if the order among the component values of each vector are all equal, then rank revarsal among the alternatives does not occur no matter how a_{kj} changes in its domain given by (12). (Proof is omitted)

Speaking reversely, this means that, when Theorem 4 is not satisfied, the order among the alternatives may change from the present order to a different order, depending upon the way how a_{kj} changes.

4. APPLICATION TO DWELLING SELECTION PROBLEM

4.1 Setting of problem

Here, the decision problem under research is a dwelling selection problem. Suppose a man (decision maker) is seeking for a dwelling house. After examining various properties appearing in a dwelling information magazine, he selected three dwellings from among them as candidatory ones (alternatives), and adopted the AHP as a tool for deciding the preference order of these three alternatives. In accordance with the procedure of the AHP, he first made a hierarchy of the dwelling selection problem as shown in Fig.1(See Note 1). Then he decided the value of the priority matrix of each level as shown by A_2^*, A_3^* and A_3^*. Further, he calculated the composite priority of the alternatives from these values, using (3), and cosequently obtained $w_4^*=(0.381\ 0.269\ 0.350)^T$. Eventually, the preference order among three alternatives became A > C > B, and the alternative A proved to be the most desirable house.

Note 1: The hierarchy shown in Fig.1, originally, was a short hierachy in which the element s_{21} of the second level was directly connected with the elements s_{41}, s_{42} and s_{43} of the fourth level. In order to convert it into the complete hierarchy (Saaty 1980), the dummy element s_{31} is inserted into the third level, and the priority values of all the elements having no direct parent-child relationship are made zero in Fig. 1, hence $a_{33}^*=(\ 0\ 0\ 0\ 0.6\ 0.1\ 0.3,)^T$, for example.

However, he feels a little anxious about this result of analysis, because among the obtained priority matrix values, he has no confidence especially in the priority vector value a^*_{33}=(0 0 0 0.6 0.1 0.3 $)^T$ of the subcriteria of the third level with respect to the element s_{23} (Residential environment) of the second level. He is very eager to know how the change of a^*_{33} will affect the result of the present analysis. The above is setting of the problem of this example.

4.2 Results of sensitivity analysis

Here, we attempt various sensitivity analyses to the change of priority vector a_{33} under the problem setting mentioned in 4.1. For this purpose, we calculate priority reachability matrix M in accordance with (5), and then obtain the value as shown in Fig.2.

First of all, let us consider the case where a_{33} changes a little from the present value a^*_{33} to a new value $a^*_{33}+\Delta a_{33}$=(0 0 0 0.5 0.15 0.35 $)^T$. In this case, Δa_{33} = (0 0 0 -0.1 0.05 0.05 $)^T$ and also L=4, k=3, j=3, I_{33}= {4,5,6}, $n(I_{33})$=3. From Theorem 1, we can find that only the four blocks M_{31}, M_{32}, M_{41} and M_{42} are affected by the above change of a_{33}. For example, the block M_{42} changes from its present value M^*_{42} to

$$M^*_{42} + \Delta M_{42} = M^*_{42} + M^*_{43}\Delta a_{33}[M^*_{22}]_3 = \begin{bmatrix}0.5 & 0.27 & 0.25\\ 0.2 & 0.33 & 0.35\\ 0.3 & 0.40 & 0.40\end{bmatrix} + \begin{bmatrix}0 & 0 & -0.015\\ 0 & 0 & 0.005\\ 0 & 0 & 0.010\end{bmatrix} = \begin{bmatrix}0.5 & 0.27 & 0.235\\ 0.2 & 0.33 & 0.355\\ 0.3 & 0.40 & 0.410\end{bmatrix}. \quad (17)$$

On the other hand, from Theorem 2, composite priority vector w_4 of the alternatives changes from its present value w^*_4 = (0.381 0.269 0.350 $)^T$ to

$$w^*_4 + \Delta w_4 = w^*_4 + w^*_{23}M^*_{43}\Delta a_{33} = \begin{bmatrix}0.381\\ 0.269\\ 0.350\end{bmatrix} + \begin{bmatrix}-0.003\\ 0.001\\ 0.002\end{bmatrix} = \begin{bmatrix}0.378\\ 0.270\\ 0.352\end{bmatrix}. \quad (18)$$

Next, let us consider the case where a_{33} changes throughout its whole domain $\{a_{33} \mid a_{43}+a_{53}+a_{63}=1,\ a_{43}, a_{53}, a_{63} \geq 0\}$. From (14) in Theorem 3, vectors b_i, $i \in I_{33}$ become

$$\begin{aligned} b_4 &= M^*_{43}[A^*_3]_4\, w^*_2 = (\ 0.391\ 0.259\ 0.350\ )^T,\\ b_5 &= M^*_{43}[A^*_3]_5\, w^*_2 = (\ 0.351\ 0.239\ 0.410\ )^T,\\ b_6 &= M^*_{43}[A^*_3]_6\, w^*_2 = (\ 0.371\ 0.299\ 0.330\ )^T. \end{aligned} \quad (19)$$

Therefore, the mapping region $\Gamma_4(a_{33})$, in which composite priority vector w_4 of the alternatives can change, becomes as follows:

$$\Gamma_4(a_{33}) = \{ w_4 \mid w_4 = b_4a_{43}+b_5a_{53}+b_6a_{63},\ a_{43}+a_{53}+a_{63} = 1,\ a_{43}, a_{53}, a_{63} \geq 0 \}. \quad (20)$$

Moreover, since the mean value of the three vectors of (19) becomes $\bar{b}$ = (0.371 0.266 0.363 $)^T$, the value of sensitivity coefficient $\alpha(a_{33})$ in Definition 2 becomes 0.055.

Lastly, let us consider possibility of rank reversal among the alternatives due to global change of a_{33}. As evident from (19), the order among component values of the vector b_5 is different from those of b_4 and b_6, and they do not satisfy Theorem 4. Therefore, depending upon the way a_{33} changes, it is possible that rank reversal among the alternatives may occur. For instance, the present value of w_4 is $w_4^* = (0.378\ 0.270\ 0.352)^T$, but due to the change of a_{33} from its present value a_{33}^* to $a_{33}^* + \Delta a_{33} = (0\ 0\ 0\ 0.1\ 0.8\ 0.1)^T$, it becomes $w_4^* + \Delta w_4 = (0.357\ 0.247\ 0.396)^T$ from Theorem 2. Namely, the order between the alternative A and C reverses.
Fig.3 shows the mapping regions of a_{21}, a_{33}, a_{41}, a_{42} and a_{45} in the space of the alternatives. The broken line in this figure indicates the boundary on which reversal of the order occures, and the white point shows the position of the present composite priority values w_4^* of the alternatives. On the other hand, Table 1 shows what type of rank reversal is capable of occurring for individual a_{kj}'s, and also shows the sensitivity coefficient values for all a_{kj}'s. These results show that in this example, sensitivity of the composite priorities of the alternatives is particulary high for a_{41} and a_{42}, and that depending upon the change of these priority vectors, all combinations of orders can occur as the preference order of the alternatives. On the other hand, sensitivity for a_{32} and a_{45} is low. No matter how these may change, the preference order remains as A > C > B, and rank reversal never occur.

5. CONCLUSION

In this paper, we studied sensitivity analysis relating to priority as one of sensitivity analyses in the AHP. Namely, we examined theoretically, when the priority values of subcriteria with respect to a criterion of a certain level change, how these changes will affect the priority values of criteria or alternatives of other levels as well as to what degree they will affect the said priority values, and further, whether it is capable of causing rank reversal among the alternatives. In consequence, by deriving some theorems, we clarified that all these problems can easily be solved analytically, using the priority reachability matrix newly defined in this paper. Further, by applying these theorems actually to a simple dwelling selection problem, we confirmed usefulness thereof. Finally, as one of the problems to be solved in future, we show sensitivity analysis when the criteria or the alternatives are newly added or deleted.

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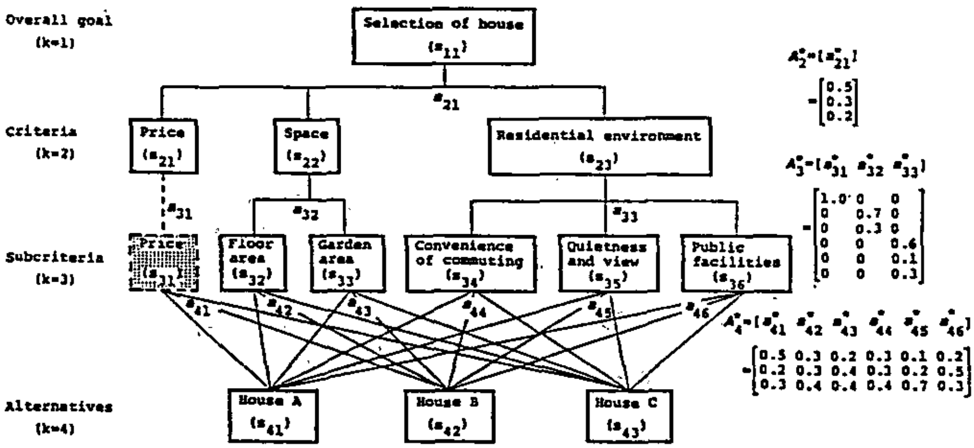

Figure 1 Hierarchy of the dwelling selection problem.

$$M^* = I + A + A^2 + A^3 = \begin{bmatrix} M_{11}^* (=w_1^*) & 0 & \cdots & 0 \\ M_{21}^* (=w_2^*) & M_{22}^* & \ddots & \vdots \\ M_{31}^* (=w_3^*) & M_{32}^* & M_{33}^* & 0 \\ M_{41}^* (=w_4^*) & M_{42}^* & M_{43}^* & M_{44}^* \end{bmatrix}, \quad \text{where} \quad A = \begin{bmatrix} 0 & \cdots & & 0 \\ A_2^* & \ddots & & \vdots \\ 0 & A_3^* & \ddots & \\ 0 & 0 & A_4^* & 0 \end{bmatrix}$$

Figure 2 Priority reachability matrix in the dwelling selection problem.

	s_{11}	s_{21}	s_{22}	s_{23}	s_{31}	s_{32}	s_{33}	s_{34}	s_{35}	s_{36}	s_{41}	s_{42}	s_{43}
s_{11}	1												
s_{21}	0.5	1	0	0									
s_{22}	0.3	0	1	0									
s_{23}	0.2	0	0	1									
s_{31}	0.50	1.0	0	0	1	0	0	0	0	0			
s_{32}	0.21	0	0.7	0	0	1	0	0	0	0			
s_{33}	0.09	0	0.3	0	0	0	1	0	0	0			
s_{34}	0.12	0	0	0.6	0	0	0	1	0	0			
s_{35}	0.02	0	0	0.1	0	0	0	0	1	0			
s_{36}	0.06	0	0	0.3	0	0	0	0	0	1			
s_{41}	0.381	0.50	0.27	0.25	0.5	0.3	0.2	0.3	0.1	0.2	1	0	0
s_{42}	0.269	0.20	0.33	0.35	0.2	0.3	0.4	0.3	0.2	0.5	0	1	0
s_{43}	0.350	0.30	0.40	0.40	0.3	0.4	0.4	0.4	0.7	0.3	0	0	1

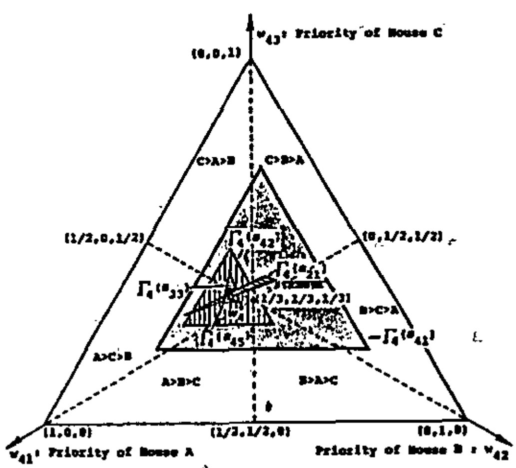

Figure 3 Mapping regions of a_{kj}'s.

Table 1 Sensitivity coefficient and possibility of rank reversal for each a_{kj}.

Ranks of alternatives A,B,C							Sensitivity coefficient
Rank 1	A	A	B	B	C	C	
Rank 2	B	C	A	C	A	B	$\alpha(a_{kj})$
Rank 3	C	B	C	A	B	A	
a_{21}	×	○	×	×	○	○	0.171
a_{32}	×	○	×	×	×	×	0.030
a_{33}	×	○	×	×	○	×	0.055
a_{41}	○	○	○	○	○	○	0.500
a_{42}	○	○	○	○	○	○	0.210
a_{43}	○	○	×	×	○	×	0.090
a_{44}	○	○	○	×	○	×	0.120
a_{45}	×	○	×	×	×	×	0.020
a_{46}	×	○	×	×	○	×	0.060