Inference in Pairwise Comparison Experiments
Based on Ratio Scales

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ABSTRACT

A multiplicative model is proposed for Saaty's method of scaling in paired comparisons experiments. Iterative schemes are given for the maximum likelihood estimation of priority weights for the alternatives under this model that converge monotonically to the maximum likelihood estimates.The asymptotic distribution of these estimates are obtained and their accuracy evaluated by a Monte Carlo study. Finally, numerical examples are used to illustrate the method.

1. INTRODUCTION

Given the task of comparing t mutually exclusive alternatives $T_1, T_2, ..., T_t$ for preference, Saaty (1977) has developed a new approach for deriving priority weights associated with these alternatives. Instead of mere preference or indifference, he introduced quantification of the level of preferences, aggregating over a number of judges who compare the alternatives in pairs.

Each judge in a group provides an observation on a pairwise comparison matrix $A = ((a_{ij}))$, where a_{ij} is a positive number indicating the intensity of preference (assumed to be on a ratio scale) of alternative T_i over T_j.
Based on a multiplicative model of the form

$$a_{ij} = \frac{\pi_i}{\pi_j} \varepsilon_{ij}, \tag{1}$$

where π_i and π_j are parameters denoting priority weights of alternatives T_i and T_j, and ε_{ij} is an appropriately specified error term, Saaty (1980) estimates priority weights by the principal right eigenvector of the matrix A. He also assumes that the matrix A is reciprocal, i.e., $a_{ji} = 1/a_{ij}$ for all i, j.

Since the appearance of the eigenvalue method in Saaty (1977), various other scaling techniques have been proposed by several authors, notably least squares (Jensen, 1984) and logarithmic least squares (Rabinowitz, 1976; Saaty, 1980; DeGraan, 1980; Fitchner, 1983; De Jong, 1984; Williams and Crawford; 1985). In these methods, the multiplicative model is transformed to a log-linear form :

$$\ln a_{ij} = \ln \pi_i - \ln \pi_j + \ln \varepsilon_{ij}, \tag{2}$$

where the expected value of $\ln \varepsilon_{ij}$ is assumed to be zero. In terms of the original multiplicative model (1), however, this assumption implies that the expected value of a_{ij} cannot be π_i / π_j. This is because the expected value of ε_{ij} cannot be unity if the expected value of $\ln \varepsilon_{ij}$ is zero (see appendix A). To avoid this anomaly, we work with the original multiplicative model (1).

We develop the multiplicative model directly in section 2 along with some additional assumptions. Although the usual practice is to collect only the entries above the diagonal (given the reciprocal matrix where $a_{ji} = 1/a_{ij}$), we will consider collecting both the entries a_{ij} and a_{ji} for more information and to increase the number of observations. Also in that case the estimation problem is invariant to the indexing of the items because of the fixed entries in both a_{ij} and a_{ji}. For example, if one observes a_{12} to be 3, then if some time gap is allowed before observing a_{21}, the latter may be observed at some value other than 1/3 even for the same individual. This is not surprising given the existence of errors, represented by ε_{ij} in equation (1). A virtue of the model developed here is that the error contributions are treated naturally via maximum likelihood procedures. Moreover, by asking the same questions two ways, increased information for estimating the true preference is obtained. Of course, the derived scale values will yield a reciprocal matrix.

In section 3 the maximum likelihood estimates (mles) of the priority weights are found using iterative schemes that are shown to be monotonically convergent. These estimates synthesize the judgments of the individuals in a particular group. The asymptotic distribution of the estimator is obtained and a Monte Carlo study is included to evaluate the accuracy of the asymptotic distribution in the same section. Numerical examples are presented to illustrate the method in section 4.

2. THE MATHEMATICAL MODEL

Consider a paired comparison experiment with t objects, $T_1, T_2, \ldots, T_t$. Let π_i be the priority weight of T_i, $\pi_i \geq 0$ and $\Sigma\, \pi_i = 1$ (ensuring determinacy). For a particular T_i and T_j; $i \neq j$, the preference of T_i over T_j is given by N independent judges. We assume that each individual gives a judgment on just one preference problem, requiring a total of Nt(t-1) judges. This procedure guarantees independence of observations, but it does entail the implicit assumption that individual differences among the judges are unsystematic and can be ignored. Thus, the judge giving the score a_{ijk} is identified not by k alone but by the ordered triplet (i,j,k) of the scripts. The following multiplicative model is considered :

$$a_{ijk} = \frac{\pi_i}{\pi_j}\, \varepsilon_{ijk}\,. \tag{3}$$

The score a_{ijk} assigned by the k-th judge to T_i over T_j may be thought of as the product of two components . The first , $\dfrac{\pi_i}{\pi_j}$, is the average preference of T_i over T_j in the population to which the k-th judge belongs. The second, ε_{ijk}, represents the deviation of the k-th judge from this average preference. Only one set of scale values $\pi_1, \ldots, \pi_t$ will be derived to represent the population of the judges leaving the systematic individual differences to the error values ε_{ijk}. In the model (3) ε_{ijk}s are assumed to be independent and do not depend on π_js.

It is desirable that the expected value of a_{ijk}, $E[a_{ijk}]$ be $\dfrac{\pi_i}{\pi_j}$. Hence we assume that $E[\varepsilon_{ijk}] = 1$. We assume ε_{ijk}s to be identically distributed as Gamma (r,r) since these are positive random variables and the choice of the Gamma distribution for judgment matrices have been shown to be appropriate by Vargas (1982). One can use this assumption if the data fit the Gamma (r, r) distribution. r is treated as a parameter and is estimated jointly with the parameters $\pi = (\pi_1, \pi_2, .. ,\pi_t)$. In section 3 we observe that the estimation of priority weights $\pi_1, \pi_2, ..., \pi_t$ does not depend on r . On the other hand, the estimate of r depends on the priority weights. The mles of $\pi_1, \pi_2, ..., \pi_t$ derived in section 3 can be used to obtain the mle of r as a solution to the following equation :

$$\psi(r) - \ln(r) = 1 + \frac{1}{NT} \left[\sum_{i \neq j} \sum_{k} b_{ijk} - \sum_{i \neq j} \frac{\pi_j}{\pi_i} \sum_{k} a_{ijk} \right] \quad (4)$$

where $b_{ijk} = \ln a_{ijk}$, $T = t(t-1)$, $\psi(r)$ = digamma function = $\frac{d \ln\Gamma(r)}{dr}$.

(See appendix B).

For the rest of the paper we assume that ε_{ijk}s are independently distributed as Gamma (r, r), where r is obtained from (4).

Next , we test the validity of the model (3) by the null hypothesis $H_0 : \pi_{ij} = \frac{\pi_i}{\pi_j}$; against the general alternative hypothesis $H : \pi_{ij} \neq \frac{\pi_i}{\pi_j}$ for some i, j for the general model $a_{ijk} = \pi_{ij}\, \varepsilon_{ijk}$.

Denoting λ as the likelihood ratio statistic, $-2\ln\lambda$ given by

$$-2r \sum_{k=1}^{N} \sum_{i \neq j} \frac{a_{ijk}}{p_{ij}} - 2\,N\,r \sum_{i \neq j} \ln p_{ij} + 2r \sum_{k=1}^{N} \sum_{i \neq j} \frac{a_{ijk}\, p_j}{p_i}$$

has the χ^2 distribution, for large N with (t-1)(t-2)/2 df under H_0 where p_i represents mle of π_i under the restriction $p_1 + ... + p_t = 1$ and p_{ij} represents mle for π_{ij} . In the event the model (3) is accepted, the priority weights in that model are estimated.

3. Estimation Of The Priority Weights

The priority weights of the elements are estimated in this section. The logarithm of the likelihood function $\ln L(\pi)$ apart from an additive constant is

$$-\sum_{k=1}^{N} \sum_{i \neq j} r\, a_{ijk} \frac{\pi_j}{\pi_i} \quad (5)$$

The mle p of π is obtained as solution to the following system of equations subject to the constraint $\Sigma \pi_i = 1$.

$$\frac{1}{p_i^4} \left[g_i(p) - p_i^4 S_i \right] = 0 \quad (6)$$

where $g_i(p) = p_i^2 \sum_{j \neq i} \sum_{k=1}^{N} a_{ijk}\, p_j$ and $S_i = \sum_{j \neq i} \sum_{k=1}^{N} a_{jik} \frac{1}{p_j}$.

We can readily see that L (π) is positive in the region $\{\pi_i > 0 ;\ \Sigma\ \pi_i = 1\}$. The existence of the maximum is established by the fact that defining L (π) for π on the boundary gives continuous extension of L(π) to the closed region $\{\pi_i \geq 0; \Sigma\pi_i=1\}$. Adapting the technique of Ford [1957] and subsequently of Davidson [1970], we give the proposed iterative schemes for obtaining solutions to the previous equations in the following.

Each stage of the iteration is indexed by K, K=1,2,... . For each value of K, a revised value of each p_i is obtained. A stage is sub-divided into sub-stages indexed by n; n = (K-1)s,..., Ks-1. For each sub-stage, a new estimate p of π is obtained through change of one element of p at a time. The (n+1)-th sub-stage value $p^{(n+1)}$ is obtained from the n-th sub-stage value $p^{(n)}$ through replacement only of the element $p_i^{(n)}$ for which i = n - (K-1)s+ 1. Finally, the iterative scheme is given as follows.

$$[p_i^{(n+1)}]^4 = \frac{g_i\,(p^{(n)})}{S_i^{(n)}} \tag{7}$$

From the iterative scheme, it is clear that one gets final estimates positive, if one starts with positive initial estimates. As initial estimates, one may use $p_i^{(0)} = 1/t$; i=1,2,..., t.We show in Appendix C that solutions to the iterative scheme given in (7) converge monotonically to the solution of (6).

Using large sample theory, it can be shown that $[\sqrt{N}\,(p_1 - \pi_1), \ldots, \sqrt{N}\,(p_t - \pi_t)]'$ has singular normal distribution of dimensionality (t −1) in a space of t dimensions with mean vector zero and disperson matrix Σ, which is given by

$$\Sigma = \begin{pmatrix} \Sigma_1 & b \\ b' & \sigma_{oo} \end{pmatrix}, \tag{8}$$

where $\Sigma_1 = ((\sigma_{ij})) = ((C_{ij}))^{-1}$,

$$C_{ii} = \frac{2Nr(t-1)}{\pi_i^2} + \frac{2Nr(t-1)}{\pi_t^2} + \frac{2Nr}{\pi_i\pi_t}, \quad i = 1, 2, \ldots, t-1,$$

$$C_{ij} = \frac{2Nr(t-1)}{\pi_t^2} + \frac{2Nr}{\pi_i\pi_t} + \frac{2Nr}{\pi_j\pi_t} - \frac{2Nr}{\pi_i\pi_j}; \quad i, j = 1, 2, \ldots, t-1 ;\ i \neq j$$

$$b' = (b_1, \ldots, b_{t-1}) \text{ where } b_i = -\sum_{j=1}^{t-1} \sigma_{ij}\ ; i=1,2,\ldots,t-1 \text{ and } \sigma_{oo} = \sum_{i=1}^{t-1}\sum_{j=1}^{t-1} \sigma_{ij}.$$

The proof of the above is sketched in Appendix D.

An approximate confidence interval for π_i; i=1,2,...,t, a confidence region for any subset of s distinct parameters of the set, s < t and a confidence interval for $\ln \pi_i$; i =1,2,...,t can be obtained using the distribution given by (8).

It only remains to be checked how large N and t must be for the approximation to be satisfactory. For this purpose, we simulate N pairwise comparison matrices A, whose (i, j)th entry a_{ij} is obtained through (3) starting with a known vector π and assuming that ε_{ij}s are distributed as Gamma (r, r) distribution with r estimated through (4). The estimate p of π is obtained from these N generated matrices. This process is carried out n times yielding n estimates $p^{(1)}, p^{(2)}, \dots, p^{(n)}$. Based on these, the normality of the estimate is tested using a test criterion given by Bera and John (1983). The test criterion is

$$C = n \sum_{i=1}^{t} T_i^2/6,$$

where
$$T_i = \sum_{j=1}^{n} \frac{[y_i^{(j)}]^3}{n},$$

$$y^{(j)} = R \cdot (p^{(j)} - \bar{p}),$$

$y_i^{(j)}$ is the i-th component of the vector $y^{(j)}$, $\bar{p}$ is the vector of the sample means of the estimate vectors $p^{(1)}, p^{(2)}, \dots, p^{(n)}$ and

$$R = (1/\sqrt{\lambda_1})P_1P'_1 + (1/\sqrt{\lambda_2})P_2P'_2 + \dots + (1/\sqrt{\lambda_{t-1}})P_{t-1}P'_{t-1} \,..$$

Here, λ's are the eigenvalues and P's are the corresponding eigenvectors of S, the sample covariance matrix of $p^{(j)}$s. The test criterion C has asymptotically the χ^2 distribution with t d.f. under H_0.

We carried out this simulation process for t = 3, 4, 10 and N = 3, 5, 10 and 20 for each value of t. Throughout the study we used n = 1000. We summarise the simulation result in Table 1. For t = 3, 4, 10, the fixed vector π was chosen respectively as follows:

$\pi = (0.2, 0.3, 0.5)$
$\pi = (0.1, 0.4, 0.3, 0.2)$
$\pi = (0.1, 0.1, 0.1, 0.05, 0.05, 0.05, 0.15, 0.15, 0.15, 0.1)$

Insert Table 1 about here

Comparing these values with the corresponding critical values, it can be seen that for the values t and N as small as 3, the asymptotic property holds well.

4. Numerical Examples

In this section, we present hypothetical data on the first level of the school selection example given in Saaty (1980) to illustrate the method developed above. Seven independent observations for each of the thirty pairs of six criteria – Learning(L), Friends(F), School life(S), Vocational training(V), College preparation(C), Music classes(M), were collected from judges of similar background in an hypothetical experiment. The data are recorded in Table 2. For example, the fourth observation on the pair (L,F) is 1.5 which means that L is preferred 1.5 times F by the fourth judge.

Insert table 2 about here

The iterative scheme (7) converged at the fourth iteration yielding the estimate as :

(0.29, 0.16, 0.05, 0.13, 0.22, 0.15)

whereas the estimate of De Jong(1984) turns out to be

(0.33, 0.15, 0.05, 0.14, 0.19, 0.14).

We have constructed the hypothetical data in table 2 as random fluctuations around the example given in Saaty (1980) collected from a single individual. We observe that the estimates not only maintain the same ranking of the priority weights, the values are also very close to the solution of Saaty's Eigenvalue method. The estimated asymptotic dispersion matrix of p (obtained from (8)) is given by :

$$\begin{pmatrix} 0.00287 & -0.00089 & -0.00011 & -0.00051 & -0.00101 & -0.00035 \\ -0.00089 & 0.00117 & 0.00001 & -0.00005 & -0.00018 & -0.00006 \\ -0.00011 & 0.00001 & 0.00006 & 0.00002 & 0.00001 & 0.00001 \\ -0.00051 & -0.00005 & 0.00002 & 0.00062 & -0.00007 & 0.00001 \\ -0.00101 & -0.00018 & 0.00001 & -0.00007 & 0.00141 & -0.00016 \\ -0.00035 & -0.00006 & 0.00001 & 0.00001 & -0.00016 & 0.00023 \end{pmatrix}$$

Next, we consider a similar data set but with larger variance to show how the bias increases significantly in log-linear form. These data are recorded in table 3. The interpretation is similar to that of table 2.

Insert table 3 about here

The iterative scheme converged at the sixth iteration with the estimate as :

(0.29, 0.15, 0.05, 0.14, 0.21, 0.16)

whereas the estimate of De Jong(1984) is given by

(0.32, 0.15, 0.04, 0.12, 0.17, 0.20)

We observe that the estimates obtained in our method maintain the same ranking of the priority weights and the values are also very close to the solution of Saaty's Eigenvalue method. However, in De Jong's method the ranks of the fifth and sixth object are reversed.

5. Concluding Remarks

In this paper we concentrate our attention on estimates of priority weights with respect to a fixed criterion. However, the procedure could be easily extended to the situation of more than one criterion in the analytic hierarchy process set up.

For each of the t(t-1) questions of preference, N independent observations are obtained from N different judges. Only one set of scale values is derived to represent a group of judges - disregarding the individual differences among members of that group. We assume that a particular group is homogeneous and obtain what the individuals are offering as a group. It has been pointed out that it may be inconvenient to use several judges. From a practical stand point this is not a serious problem as we find from the Monte Carlo study that with three alternatives, as few as three judges for each of the six pairs would be sufficient.

Once a set of scale values is derived we obtain a reciprocal matrix rather than forcing the reciprocity before the model's application. Collection of all off-diagonal entries does not violate the rationality of making comparisons. It merely increases the information regarding the true preferences by asking the same question in two different ways.

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APPENDIX A

Lemma : If lnX is distributed as Normal with expected value zero and variance σ^2 then E(X) cannot be 1. Conversely, if E(X) is 1 then the expected value of lnX can not be zero.

Proof : If lnX is distributed as Normal with expected value zero and variance σ^2, then X is said to be log-normally distributed. In that case,

$$E(X) = \exp(\sigma^2/2).$$

Therefore, $E(X) = 1$ implies $\sigma^2 = 0$ which is impossible.
Again $E(X) = 1$ implies $\exp(\mu + \sigma^2/2) = 1$ where μ is the expected value of lnX. It follows that $\mu = -\sigma^2/2$ which is never zero. This completes the proof.

Remark : In general, $E(\ln X) < \ln E(X)$ since the function "−ln" is a convex function. Therefore, $E(\ln X) < 0$ whenever $E(X)$ is 1.

APPENDIX B

Lemma : The maximum likelihood estimate of r is obtained as solution to the equation :

$$\psi(r) - \ln(r) = 1 + \frac{1}{NT}\left[\sum_{i\neq j}\sum_k b_{ijk} - \sum_{i\neq j}\frac{\pi_j}{\pi_i}\sum_k a_{ijk}\right] \qquad \text{(A1)}$$

where $b_{ijk} = \ln a_{ijk}$, $T = t(t-1)$ and $\psi(r)$ = digamma function = $\frac{d\ln\Gamma(r)}{dr}$.

Proof : In the model (3), ε_{ijk} is assumed to have Gamma (r, r) distribution. The probability density of $w_{ijk} = \ln(r.\ \varepsilon_{ijk})$ is then given by

$$f(w_{ijk}) = \frac{e^{r\,w_{ijk}} - e^{w_{ijk}}}{\Gamma(r)}.$$

The logarithm of the likelihood function, lnL, after some simplifications reduces to :

$$NT.\ r.\ \ln(r) + r\sum_{i\neq j}\sum_k b_{ijk} - r\sum_{i\neq j}\frac{\pi_j}{\pi_i}\sum_k a_{ijk} - NT\,\Gamma(r)$$

Differentiating the above expression with respect to r, one obtains

$$\frac{\partial \ln L}{\partial r} = NT.\ln(r) + NT + \sum_{i<j}\sum_k b_{ijk} - \sum_{i<j}\frac{\pi_j}{\pi_i}\sum_k a_{ijk} - NT\,\Psi(r)$$

Setting this derivative equal to zero and dividing both sides by NT, (A1) is obtained.

APPENDIX C

Lemma : ln L is increased at a substage if and only if the estimated value at that substage is changed.

Proof : Let $\left.\frac{\partial \ln L}{\partial \pi_i}\right|_n$ denotes the value of $\frac{\partial \ln L}{\partial \pi_i}$ when π is replaced by $p^{(n)}$.

Let $\left.\frac{\partial \ln L}{\partial \pi_i}\right|_{\ln}$ denotes the value of $\frac{\partial \ln L}{\partial \pi_i}$ when π is replaced by $p^{(n)}$.

It follows from (6) and (7) that

$$\frac{\partial \ln L}{\partial \pi_i}\Big|_n = \frac{1}{[p_i^{(n)}]^4}\{ g_i(p^{(n)}) - [p_i^{(n)}]^4 \cdot S_i^{(n)} \}$$

$$= \frac{1}{[p_i^{(n)}]^4}\{ [p_i^{(n+1)}]^4 \cdot S_i^{(n)} - [p_i^{(n)}]^4 \cdot S_i^{(n)} \}$$

$$= \frac{S_i^{(n)} \cdot \Delta p_i \cdot [p_i^{(n+1)} + p_i^{(n)}]\ [(p_i^{(n+1)})^2 + (p_i^{(n)})^2]}{[p_i^{(n)}]^4}$$

where $\Delta p_i = p_i^{(n+1)} - p_i^{(n)}$ so that Δp_i has the same sign as $\frac{\partial \ln L}{\partial \pi_i}\Big|_n$. Again using (6) and (7) one obtains

$$\frac{\partial \ln L}{\partial \pi_i}\Big|_{n+1} = \frac{1}{[p_i^{(n+1)}]^4}\{ g_i(p^{(n+1)}) - [p_i^{(n+1)}]^4 \cdot S_i^{(n)} \}$$

$$= \frac{1}{[p_i^{(n+1)}]^4}\{ g_i(p^{(n+1)}) - g_i(p^{(n)}) \},$$

which is of the same sign as Δp_i since $g_i(p)$ is monotone increasing in p_i. Now, $\pi_i \frac{\partial \ln L}{\partial \pi_i}$ is monotone decreasing in π_i which follows from the result that

$$\frac{\partial \ln L}{\partial \pi_i} + \pi_i \cdot \frac{\partial^2 \ln L}{\partial \pi_i^2} = - \frac{\sum_{j \neq i} \sum_{k=1}^{N} a_{ijk}\pi_j}{\pi_i^2} - \sum_{j \neq i} \sum_{k=1}^{N} a_{jik} \frac{1}{\pi_j} < 0 .$$

Therefore, $\frac{\partial \ln L}{\partial \pi_i}$ has the same sign for all π_i between $p_i^{(n)}$ and $p_i^{(n+1)}$. Thus, the change in the likelihood

$$\Delta \ln L = \Delta p_i \frac{\partial \ln L}{\partial \pi_i}\Big|_\varepsilon \geq 0,$$

equality is achieved if and only if $\Delta p_i = 0$.
$\frac{\partial \ln L}{\partial \pi_i}\Big|_\varepsilon$ denotes $\frac{\partial \ln L}{\partial \pi_i}$ at $p^{(n)} + \varepsilon \Delta p_i I_i$, where, $I_i = (0, 0, \ldots, 1, 0, \ldots, 0)$

1 being at the ith position. $0 < \varepsilon < 1$.

In the above steps, it is proved that likelihood is increased at every substage if and only if the corresponding parameter value is changed, which guarantees the monotone convergence to the maximum likelihood estimate.

APPENDIX D

We find the expressions for C_{ii} and C_{ij} of (8). Let ln L be as in (5).

$$C_{ii} = E[-\frac{\partial^2 \ln L}{\partial \pi_i^2}]$$

$$= \frac{2Nr(t-1)}{\pi_i^2} + \frac{2Nr(t-1)}{\pi_t^2} + \frac{2Nr}{\pi_i \pi_t} \text{ since } E(a_{ij}) = \frac{\pi_i}{\pi_j}.$$

and

$$C_{ij} = E[-\frac{\partial^2 \ln L}{\partial \pi_j \partial \pi_i}]$$

$$= \frac{2Nr(t-1)}{\pi_t^2} + \frac{2Nr}{\pi_i \pi_t} + \frac{2Nr}{\pi_j \pi_t} - \frac{2Nr}{\pi_i \pi_j} \text{ using } E(a_{ij}) = \frac{\pi_i}{\pi_j}.$$

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Table 1 Computed value of C for t=3, 4, 10 and N=3, 5, 10, 20

t \ N	3	5	10	20
3	.1467	.1015	.0146	.0034
4	.1005	.0234	.0158	.0017
10	.0589	.0192	.0035	.0009

Table 2. Comparisons Data of six criteria with respect to overall satisfaction with school ; N = 7.

Pairs	Observations						
	1	2	3	4	5	6	7
(L,F)	4	3	2	1.5	4.5	4	4.5
(L,S)	3	2.5	2	3.2	2	2.5	3.4
(L,V)	1	1.5	0.5	1	0.5	2	1
(L,C)	3	2.5	4	3.5	3	3.5	2.5
(L,M)	4	3	3.5	3	4.5	4	5.5
(F,S)	6	5.5	8	4.5	7.5	7	6.5
(F,V)	3	2.5	3	4	3.5	2	2.5
(F,C)	0.2	0.25	0.33	0.2	0.17	0.2	0.25
(F,M)	1	1.5	1	2	1.5	0.5	2
(S,V)	0.2	0.25	0.2	0.25	0.33	0.2	0.25
(S,C)	0.17	0.25	0.20	0.33	0.5	0.2	0.33
(S,M)	0.2	0.25	0.33	0.25	0.15	0.2	0.33
(V,C)	1	0.5	2	0.5	1	1.5	2
(V,M)	0.33	0.5	0.5	0.33	0.5	1	0.5
(C,M)	3	2	2.5	3.5	3	2	3.5
(F,L)	0.2	0.25	0.5	0.5	0.67	0.33	0.25
(S,L)	0.5	0.33	0.5	0.33	0.33	0.5	0.33
(S,F)	0.2	0.25	0.33	0.2	0.17	0.17	0.17
(V,L)	0.5	0.5	3	0.5	2	0.5	0.5
(V,F)	0.5	0.33	0.5	0.2	0.25	0.5	0.33
(V,S)	4	3	4	4	3	4	3
(C,L)	0.5	0.33	0.25	0.33	0.5	0.33	0.5
(C,F)	4	3	2	4	5	5	3
(C,S)	5	3	5	2	3	4	3
(C,V)	1	2	0.5	3	0.5	0.5	0.33
(M,L)	0.2	0.5	0.25	0.25	0.25	0.5	0.33
(M,F)	1	0.5	0.5	0.33	0.5	3	0.33
(M,S)	4	3	4	3	5	4	3
(M,V)	4	3	2	4	3	1	3
(M,C)	0.33	0.33	0.5	0.25	0.33	0.5	0.25

Table 3. The second set of comparisons data of six criteria with respect to overall satisfaction with school ; N = 7.

Pairs	Observations						
	1	2	3	4	5	6	7
(L,F)	4	6	8	3	2	1	1.5
(L,S)	3	5	7	2	1	1.5	4.5
(L,V)	1	0.5	0.5	1	2	1	1
(L,C)	3	2	1	2.5	5	6	3.5
(L,M)	4	6	7	3	2	2.5	3.5
(F,S)	7	7.5	8	5	4	3	6
(F,V)	3	5	6	2	1.5	2	3.5
(F,C)	0.2	0.4	0.6	0.1	0.15	0.2	0.2
(F,M)	1	1	0.5	2	0.5	2	1
(S,V)	0.2	0.4	0.6	0.1	0.15	0.25	0.2
(S,C)	0.2	0.6	0.4	0.1	0.2	0.25	0.15
(S,M)	0.17	0.34	0.17	0.25	0.10	0.08	0.1
(V,C)	1	0.5	2	1	1	2	0.5
(V,M)	0.33	0.5	0.5	0.2	0.25	0.3	0.33
(C,M)	3	5	6	2	1	1.5	2
(F,L)	0.2	0.2	0.5	0.67	0.67	0.33	0.25
(S,L)	0.5	0.25	0.5	0.2	0.5	0.5	0.33
(S,F)	0.2	0.25	0.33	0.2	0.5	0.5	0.5
(V,L)	0.5	4	3	0.5	2	0.33	0.25
(V,F)	0.5	0.25	0.5	0.25	0.5	0.5	0.33
(V,S)	4	2	4	1	3	5	3
(C,L)	0.5	0.25	0.2	0.33	0.5	1	0.5
(C,F)	4	1	1	4	6	5	3
(C,S)	5	1	6	1	3	4	3
(C,V)	1	3	0.33	3	0.25	0.5	0.33
(M,L)	0.2	0.5	1	0.2	0.25	1	0.33
(M,F)	1	2	0.5	0.17	0.5	3	0.33
(M,S)	4	2	1	3	6	4	3
(M,V)	1	3	2	6	3	1	3
(M,C)	0.33	1	0.17	0.25	0.33	0.5	0.25