Analysis of AHP by BIBD

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ABSTRACT

The essence of AHP is to evaluate objects in terms of the eigen vector of the comparison matrix. But when the number of objects, n, is too large, it causes often worse reliability for an observer to evaluate all paired comparisons at a time. So it is necessary to decompose the whole set of pairs into several classes, and for each class to be evaluated by one observer. We propose the decomposition by BIBD (balanced incomplete block design) well known in the field of experimental design or combinatorics. We show by simulation experiments that our method gives better evaluations than the ordinary AHP.

In connection with these, we propose the logarithmic least square method, very easy to calculate, and show that this gives very good approximation to the eigen vector method when n is rather small, and that the former completely coincides with the latter when $n \leq 3$, surprisingly.

§1 Introduction

The essence of AHP (Analitic Hierarchy Process) is to evaluate objects in terms of the eigen vector corresponding to the maximal eigen value of a matrix whose (i, j) element is the ratio of evaluation of object i to object j [1] [2] [3]. The idea is to intend to unify local informations taken by paired comparison into a gloval information.

But when the number of objects is too large, it causes often worse reliability for a observer to evaluate all paired comparisons at a time. In such case, it is necessary to decompose the whole set of paires of objects into several blocks, and for each block to be observed by one observer. It is important how to decompose the set of pairs. We propose the decomposition by BIBD (Balanced Incomplete Block Design) well known in the field of experimental design [4] [5]. And we show that this method is to give better evaluation under certain assumptions by simulation experiments (§4). Further we propose the logarithmic least square method for our problem, and show that this gives a good approximation to the eigen value method in AHP (§3).

§2 BIBD

Let $E = \{1, 2, \ldots, v\}$ be the set of objects $i = 1, 2, \ldots, v$. The number of pairs among E is ${}_vC_2 = v(v-1)/2$, and if v is large this becomes very large and an observer cannot compair all such pairs at a time. Let the set of objects which an observer can accomodate with sufficient reliability, be called an "allowable block", and let us denote the size of allowable block by k ($\leq v$).

Then we need to decompose the set of ${}_vC_2$ pairs into the classes of size ${}_kC_2$ and to allocate several observers to these classes. Each observer makes paired comparisons in his class. We unify these results and can get the evaluation on E.

For example, there are $v = 7$ applicants for a prize essay, and we try to judge their essays and to decide ranking on them. Let the allowable block size be $k = 3$, that is, one judge can read 3 essays and make paired comparisons on them. In this case, the ${}_7C_2 = 21$ pairs are decomposed into classes of size ${}_3C_2 = 3$. So we need $21/3 = 7$ judges. We unify the results of 7 judges into the whole ranking on 7 applicants.

It is the problem how to decompose the set of ${}_vC_2$ pairs into blocks and how to unify the results of observations on blocks into the whole evaluation. We propose the decomposition by BIBD and the unification by the eigen value analysis used in AHP.

BIBD on $E = \{1, 2, \ldots, v\}$ is the class $\mathrm{D} = \{B_1, \ldots, B_b\}$ of subsets (called "blocks") $B_t \subsetneqq E$ ($t = 1 \sim b$) satisfying the followings.

(i) for any $t = 1 \sim b$

$|B_t| = k$, (the block size is constant)

(ii) for any $i = 1 \sim v$

$\#\{t \mid i \in B_t, t = 1 \sim b\} = r$ (the repetition number is constant)

(iii) for any $i, j = 1 \sim v$ ($i \neq j$)

$\#\{t \mid i \in B_t, j \in B_t, t = 1 \sim b\} = \lambda$ (the intersection number is constant)

(where $|S|$ denotes the size of a set S.)

It is clear that (iii) implies (ii), so (i) and (iii) suffice for D to be BIBD. Specifically a BIBD with $\lambda = 1$ is called Steiner system and here we consider only Steiner system, which is denoted by (v, k)-D.

<u>Example 1</u> The class of subsets of $E = \{1, 2, \ldots, 7\}$ shown in Table 1 is (7, 3)-*D*. The pairs in each block are shown in write hand of the block. All such pairs construct the set of ${}_7C_2$ pairs. In other word, the set of pairs in E is decomposed into pairs in blocks.

We can represent this situation in terms of graph theory in the following way: To construct (v, k)-D is equivalent to decompose the set of edges of a complete graph with v points into complete graphs with k points. (→Fig. 1) ———

Table 1 (7, 3)-D

$B_1 =$	{1, 2, 4}	→ 12, 14, 24
$B_2 =$	{2, 3, 5}	→ 23, 25, 35
$B_3 =$	{3, 4, 6}	→ 34, 36, 46
$B_4 =$	{4, 5, 7}	→ 45, 47, 57
$B_5 =$	{5, 6, 1}	→ 56, 51, 61
$B_6 =$	{6, 7, 2}	→ 67, 62, 72
$B_7 =$	{7, 1, 3}	→ 71, 73, 13

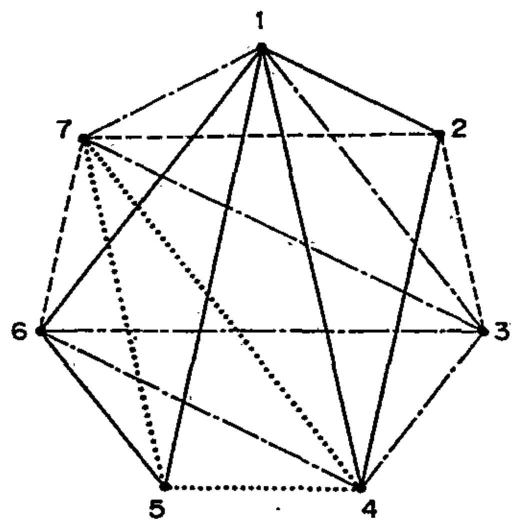

Fig. 1 BIBD decomposition

By the graph theoretic representation staded in Ex. 1 we can easily have the following relations.

$$vr = bk \tag{2・1}$$

$$b = \frac{{}_vC_2}{{}_kC_2} = \frac{v(v-1)}{k(k-1)} \tag{2・2}$$

If (v, k)-D exists then integers v and k satisfy (2. 1), (2. 2) with integers r and b, so for any integers v and k we do not necessarily have (v, k)-D. For example (8, 3)-D never exists. But (9, 3)-D exists, so we can treat the case $v=8$, $k=3$ by taking one of objects in (9, 3)-D as dummy.

The conditions of existence and construction methods of (v, k)-D have been widely and deeply researched in the field of experimental designs and combinatorial theories [4] [5].

In order to show why the decomposition by the (v, k)-D is appropriate, we propose another rather natural decomposition shown in Table 2. Of course this $D = \{B_1, \ldots, B_7\}$ is not BIBD, where pair (1, 2) occurs 2 times in B_1 and B_7, while pairs (1, 4), (1, 5) do not occur anywhere.

The author believe that (v, k)-D would give the best possible decompositions for our problems. In the end of this section we give another (v, k)-D in Table 3.

Table 2

$B_1 =$	{1, 2, 3}	→ 12, 13, 23
$B_2 =$	{2, 3, 4}	→ 23, 24, 34
$B_3 =$	{3, 4, 5}	→ 34, 35, 45
$B_4 =$	{4, 5, 6}	→ 45, 46, 56
$B_5 =$	{5, 6, 7}	→ 56, 57, 67
$B_6 =$	{6, 7, 1}	→ 67, 61, 71
$B_7 =$	{7, 1, 2}	→ 71, 72, 12

Table 3

$B_1=\{1,2,4,10\}$	→	(1, 2),	(1, 4),	(1, 10),	(2, 4),	(2, 10),	(4, 10)
$B_2=\{2,3,5,11\}$	→	(2, 3),	(2, 5),	(2, 11),	(3, 5),	(3, 11),	(5, 11)
$B_3=\{3,4,6,12\}$	→	(3, 4),	(3, 6),	(3, 12),	(4, 6),	(4, 12),	(6, 12)
$B_4=\{4,5,7,13\}$	→	(4, 5),	(4, 7),	(4, 13),	(5, 7),	(5, 13),	(7, 13)
$B_5=\{5,6,8,1\}$	→	(5, 6),	(5, 8),	(5, 1),	(6, 8),	(6, 1),	(8, 1)
$B_6=\{6,7,9,2\}$	→	(6, 7),	(6, 9),	(6, 2),	(7, 9),	(9, 2),	(9, 2)
$B_7=\{7,8,10,3\}$	→	(7, 8),	(7, 10),	(7, 3),	(8, 10),	(8, 3),	(10, 3)
$B_8=\{8,9,11,4\}$	→	(8, 9),	(8, 11),	(8, 4),	(9, 11),	(9, 4),	(11, 4)
$B_9=\{9,10,12,5\}$	→	(9, 10),	(9, 12),	(9, 5),	(10, 12),	(10, 5),	(12, 5)
$B_{10}=\{10,11,13,6\}$	→	(10, 11),	(10, 13),	(10, 6),	(11, 13),	(11, 6),	(13, 6)
$B_{11}=\{11,12,1,7\}$	→	(11, 12),	(11, 1),	(11, 7),	(12, 1),	(12, 7),	(1, 7)
$B_{12}=\{12,13,2,8\}$	→	(12, 13),	(12, 2),	(12, 8),	(13, 2),	(13, 8),	(2, 8)
$B_{13}=\{13,1,3,9\}$	→	(13, 1),	(13, 3),	(13, 9),	(1, 3),	(1, 9),	(3, 9)

§3 LLS, AHP estimation

Let $\{1, \cdots, n\}$ be the set of ofjects $i=1,2,\cdots,n$. An observer observes the ratio of evaluation of object i to object j and let us denote the observation by x_{ij}.

We assume the statistical model of x_{ij} to be

$$x_{ij}=a_{ij}\cdot e_{ij}, \quad a_{ij}=w_i/w_j \qquad (i<j,\ i,j=1\sim n)$$
$$x_{ji}=b/x_{ij} \tag{3・1}$$

Where w_i (>0) is the real evaluation of object i and is an unknown parameter, and e_{ij} (>0) is an idependent random variable representing the error of the observation. And we always recognize that any multiples of $[w_1\cdots w_n]$ are equivalent to $[w_1\cdots w_n]$ itself. Further we assume that

$$E\,(\ln e_{ij})=0, \qquad V\,(\ln e_{ij})=\sigma^2(n) \tag{3・2}$$

and $\sigma^2(n)$ is a monotone increasing function of n, the number of objects to be observed.

Whether these assumptions are reasonable or not is a psychological or a physiological problem, but we can agree with these assumptions as a trial scheme.

The main purpose of AHP is to get estimates $\hat{w}_i$ of w_i ($i=1\sim n$)by calculating the eigen vector $\hat{w}=\{\hat{w}_1, \cdots, \hat{w}_n\}$corresponding to the maximal eigen valne λ of the nxn comparison matrix $X=\{x_{ij}\}$.

Of course we have other estimation methods. The most natural one is "logarithmic least square (LLS)". For simplicity let $\bar{x}_{ij}=\ln x_{ij}$, $\bar{w}_i=\ln w_i$ and $\bar{e}_{ij}=\ln e_{ij}$. Then we have

$$\bar{x}_{ij}=\bar{w}_i-\bar{w}_j+\bar{e}_{ij} \quad (i<j,\ i,j=1\sim n) \tag{3・3}$$

Appling the least square method to(3・3)we have least last square estimate $\hat{\bar{w}}_i$ of $\bar{w}_i$, and taking inverse transform we have
$\hat{w}_i=e^{\hat{\bar{w}}_i}$ ($i=1\sim n$). This is LLS estimation.

For example let n=3. Then we have

$$\bar{x}_{12}=\bar{w}_1-\bar{w}_2+\bar{e}_{12}, \ \bar{x}_{13}=\bar{w}_1-\bar{w}_3+\bar{e}_{13}, \ \bar{x}_{23}=\bar{w}_2-\bar{w}_3+\bar{e}_{23} \tag{3・4}$$

As the vector $w=\{w_1, \cdots, w_n\}$ multiplied by an arbitral constant is equinalent to w itself, We can assume $w_1w_2w_3=1$, so we have

$$\bar{w}_1+\bar{w}_2+\bar{w}_3=0 \tag{3・5}$$

Appling least square method to(3・4),(3・5)and taking inverse transform we have

$$\hat{w}_1=(x_{12}\,x_{13})^{1/3}, \quad \hat{w}_2=(x_{23}\,x_{21})^{1/3}, \quad \hat{w}_3=(x_{32}\,x_{32})^{1/3} \tag{3・6}$$

This estimation is very simple, but we have a surprising fact that w_i ($i=1-3$) in (3・6) coincide with the components of the eigenvector corresponding to the maximal eigenvalue of the comparison matrix $X=\{x_{ij}\}$. That is, we have

<u>Theorem 1</u> In the case $n \leqq 3$, LLS estimates coincide with AHP estimates.

<u>Proof</u> In case of $n=2$, we have easily

$$\hat{w}_1=\sqrt{x_{12}}, \quad \hat{w}_2=\sqrt{x_{21}} \tag{3・7}$$

as the LLS estimates of w_1, w_2 respectively. And by direct calaculation we have

$$X \begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \end{bmatrix} = 2 \begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \end{bmatrix} \qquad X= \begin{bmatrix} 1 & x_{12} \\ x_{21} & 1 \end{bmatrix}$$

and the maximal eigen value of 2×2 comparison matrix is 2, so $\hat{w}_1, \hat{w}_2$ in (7) are the AHP estimates.

In case of $n=3$, from (6) we have

$$\begin{bmatrix} 1 & x_{12} & x_{13} \\ x_{21} & 1 & x_{23} \\ x_{31} & x_{32} & 1 \end{bmatrix} \begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{w}_3 \end{bmatrix} = \lambda \begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \\ \hat{w}_3 \end{bmatrix}$$

$$\lambda=1+r+1/r, \; r=\sqrt[3]{x_{12}\,x_{23}\,x_{31}}$$

by direct calculation. But from Perron & Frobenius Theory we can state that if a positive matrix has a positive eigen vector then it corresponds to the maximal eigen value. (A positive matrix (vetor) means a matrix (vector) whose components are all positive). (See the proof of the theorem 4 in [6]).

Like the case $n=3$ we have general LLS estimates

$$\hat{w}_i=(\Pi_{j=1}^{n} x_{ij})^{1/n} \qquad i=1, \cdots, n \tag{3・8}$$

which can be stated simply as $\hat{w}_i$ is the geometric mean of i-th row of the comparison matrix X.

But unfortunately they no longer coincide with AHP estimates for $n>3$. Thus if $n>3$ then Theorem 1 does not hold. But through Theorem 1 we can state that if $\sigma^2(n)$ is reasonably small LLS estimates must be good approximations to AHP estimates even if n is greater than 3. This also teaches us that our statistical model (3・1) is valid for the AHP analysis.

§4 Decomposition Methods by BIBD

Now we propose our decomposition methods by BIBD. We are given the set of objects $E=\{1, 2, \cdots, v\}$ to be evaluated. Let the allowable block size k be far smaller than v. We decompose E into blocks $B_1, \cdots, B_b$ which construct Steiner system (v, k)-D.

<u>Step 1</u> For each $B_e=\{\beta_1, \beta_2, \cdots, \beta_k\} \subseteq E$ an observer observes the objects and gets observation $x_{\beta_t\beta_s}$ by a paired comparison β_t to β_s ($t<s$; $t, s=1, \cdots, k$). Note that the observation error of $x_{\beta_t\beta_s}$ (measured by $V(\ln x_{\beta_t\beta_s})=\sigma^2(k)$) is far smaller than the one incurred by the observation in the whole set E.

<u>Step 2</u> Let

$$S_e=\{x_{\beta_1\beta_2}, x_{\beta_1\beta_3}, \cdots, x_{\beta_{k-1}\beta_k}\} \tag{4・1}$$

and let

$$\begin{aligned} S_e' &= \{x_{\beta_2\beta_1}, x_{\beta_3\beta_1}, \cdots, x_{\beta_k\beta_{k-1}}\} \\ &= \{1/x_{\beta_1\beta_2}, 1/x_{\beta_1\beta_3}, \cdots, 1/x_{\beta_{k-1}\beta_k}\} \end{aligned} \tag{4・2}$$

for $e=1, 2, \cdots, b$.

Then by the properties of BIBD

$$(S_1 \cup S_1') \cup (S_2 \cup S_2') \cup \cdots \cup (S_b \cup S_b')$$

constructs the set of all paired comparisons x_{ij} $(i < j,\ i, j = 1 \sim v)$, from which we construct the $v \times v$ comparison matrix $X = [x_{ij}]$

Step 3 We apply the usual AHP to the comparison matrix X, that is, we calculate the eigen vector

$\hat{w} = [\hat{w}_1, \hat{w}_2, \ldots, \hat{w}_v]$

corresponding the maximal eigen value of X. The $\hat{w}_i$ is the desired estimate of w_i in (3・1).

Example 3 (v=7, k=3, BIBD decomposition)

Let $w_1, \cdots, w_7$ (3・1) take the values shown in Table 4. Of course these are unknown for the observers and are to be estimated. Further $a_{ij} = w_i/w_j$ $(i, j = 1 \sim 7)$ are also shown in Table 4.

For each block B_e (shown in Table 2) an observer takes observations $x_{\alpha_1\alpha_2}, x_{\alpha_1\alpha_3}, x_{\alpha_2\alpha_3}$, where

$$x_{\alpha_\zeta\alpha_\Delta} = a_{\alpha_\zeta\alpha_\Delta} e_{\alpha_\zeta\alpha_\Delta} \qquad (4 \cdot 3)$$

and $e_{\alpha_\zeta\alpha_\Delta}$ is a random number whose logarism ln $e_{\beta_\zeta\beta_\Delta}$ normally distributes with zero mean and variance $\sigma^2(3)$ by our model (3・1), (3・2) (For the actual value of $\sigma^2(3)$ see (4・5)). For $e = 1, 2, \cdots, 7$ $S_e = [x_{\alpha_1\alpha_2}, x_{\alpha_1\alpha_3}, x_{\beta_2\beta_3}]$ and $S_e' = [x_{\alpha_2\alpha_1} = 1/x_{\alpha_1\alpha_2}, x_{\alpha_3\alpha_1} = 1/x_{\alpha_1\alpha_3}, x_{\alpha_3\alpha_2} = 1/x_{\alpha_2\alpha_3}]$ are shown in Table5 in the matrix form

$$X_e = \begin{bmatrix} 1 & x_{\alpha_1\alpha_2} & x_{\alpha_1\alpha_3} \\ x_{\alpha_2\alpha_1} & 1 & x_{\alpha_2\alpha_3} \\ x_{\alpha_3\alpha_1} & x_{\alpha_3\alpha_2} & 1 \end{bmatrix} \qquad (4 \cdot 4)$$

Unifying $X_1, \cdots, X_7$ we have the 7×7 comparison matrix X shown in Table 6, and calcalenlating the eigen vector for the maximal eigen valuen λ of X we have the estimates $\hat{w}_1, \cdots, \hat{w}_7$ shown also in Table 6. Comparing $\hat{w}_i$ to w_i $(i = 1 \sim 7)$ we can say that we have generally fairly good estimates.

Now we consider the actual value of $\sigma^2(n)$ in our model (3・1). Of course this depends on the given real problem. But we assum

$$\sigma(3)^2 = 0.158^2,\ \sigma^2(4) = 0.250^2,\ \sigma^2(7) = 0.5^2,\ \sigma^2(13) = 0.980^2 \qquad (4 \cdot 5)$$

as trial values in our simulations, where $\sigma^2(n)$ is roughly propotinal to ${}_nC_2$.

Example 4 (v=7, direct method)

Here we will describe the usual AHP method applied directly to Table 4. We multply a_{ij} by a random number e_{ij} and have

$x_{ij} = a_{ij} e_{ij}$ $(i < j,\ i, j = 1, \cdots, 7)$

where lne_{ij} normally distributes with zero mean and variance $\sigma^2(7) = 0.5$ (→(4.5)), and calculate $x_{ji} = 1/x_{ij}$, then we have the comparison matrix $X = [x_{ij}]$ shown in Table 7. Calculation the eigen vector for the maximal eigen value λ we have estimetes $\hat{w}_i$ $(i = 1, \cdots, 7)$ shown alson in Table 7, quite worse than the ones in Table 6.

Next we try to investigate another case $v = 13$, $k = 4$ in Examples 5, 6 along the same line as above.

Example 5 ($v = 13$, $k = 4$, BIBD decomposition)

We show w_i $(i = 1 \sim 13)$ and $a_{ij} = w_i/w_j$ $(i, j = 1 \sim 13)$ in Table 8, and(13, 4)−D and its comparison abservations in Table 9, where the logarism of the random menbers have the variance $\sigma^2(4) = 0.250^2$ (in (4・5)). Fimally the unified comparison matrix X and its maximal eigen value and the corresponding eigen vector are shown in Table 10. These give us rather good estimates.

Table 4 7×7 $[a_{ij}]$

		1	2	3	4	5	6	7
$w_1=1$	1	1	1.857	20.43	8.094	8.094	3.605	4.516
$w_2=0.538$	2	0.538	1	11.00	4.358	4.358	1.941	2.432
$w_3=0.049$	3	0.04895	0.09091	1	0.3962	0.3962	0.1765	0.2211
$w_4=0.124$	4	0.1235	0.2294	2.524	1	1	0.4154	0.5579
$w_5=0.124$	5	0.1235	0.2294	2.524	1	1	0.4454	0.5579
$w_6=0.277$	6	0.2774	0.5152	5.667	2.245	2.245	1	1.253
$w_7=0.221$	7	0.2214	0.4113	1.521	1.792	1.792	0.7983	1

Table 5 X_t X

$B_1 = \{1, 2, 4\}$, $X_1 =$

	1	2	4
1	1	1.540	7.236
2	0.6493	1	4.269
4	0.138	0.2342	1

$B_2 = \{2, 3, 5\}$, $X_2 =$

	2	3	5
2	1	13.05	6.171
3	0.07665	1	0.4057
5	0.1621	2.465	1

$B_3 = \{3, 4, 6\}$, $X_3 =$

	3	4	6
3	1	0.4033	0.1975
4	2.479	1	0.3329
6	5.063	3.004	1

$B_4 = \{4, 5, 7\}$, $X_4 =$

	4	5	7
4	1	1.030	0.5590
5	0.9709	1	0.5541
7	1.789	1.804	1

$B_5 = \{5, 6, 1\}$, $X_5 =$

	5	6	1
5	1	0.3887	0.1135
6	2.572	1	0.2632
1	8.814	3.800	1

$B_7 = \{7, 1, 3\}$, $X_7 =$

	7	1	3
7	1	0.1870	4.361
1	5.347	1	19.33
3	0.2293	0.05173	1

$B_6 = \{6, 7, 2\}$, $X_6 =$

	6	7	2
6	1	1.226	0.5317
7	0.8160	1	0.3623
2	1.881	2.760	1

Table 6 7×7 $X = \{x_{ij}\}$ (BIBD decomposition)

	1	2	3	4	5	6	7	$\lambda = 7.0291$
1	1	1.540	19.33	7.236	8.814	3.800	5.347	$\hat{w}_1 = 1$
2	0.6493	1	13.046	4.269	6.171	1.881	2.760	$\hat{w}_2 = 0.6067$
3	0.05173	0.07665	1	0.4033	0.4057	0.1975	0.2293	$\hat{w}_3 = 0.0498$
4	0.1382	0.2342	2.479	1	1.030	0.1129	0.5590	$\hat{w}_4 = 0.1222$
5	0.1135	0.1621	2.465	0.9709	1	0.3887	0.5544	$\hat{w}_5 = 0.1136$
6	0.2632	0.5317	5.063	3.004	2.572	1	1.226	$\hat{w}_6 = 0.2916$
7	0.1870	0.3623	4.361	1.789	1.804	0.8160	1	$\hat{w}_7 = 0.2132$

Table 7 $X = [x_{ij}]$ (usual AHP)

1	2	3	4	5	6	7	$\lambda = 7.323$
1	1.731	13.29	7.942	8.944	1.437	4.561	$\hat{w}_1 = 1$
0.5777	1	11.68	4.720	6.245	2.188	4.173	$\hat{w}_2 = 0.7679$
0.07524	0.08562	1	0.3694	1.190	0.1597	0.3299	$\hat{w}_3 = 0.0793$
0.1259	0.2119	0.707	1	0.5543	0.3720	0.3892	$\hat{w}_4 = 0.1303$
0.1118	0.1601	0.8402	1.801	1	0.5835	0.6611	$\hat{w}_5 = 0.1460$
0.6959	0.4570	6.262	2.688	1.714	1	2.150	$\hat{w}_6 = 0.4139$
0.2193	0.2396	3.031	2.569	1.513	0.4651	1	$\hat{w}_7 = 0.2201$

Example 6 ($v = 13$, direct method)

If an observer observes ${}_{13}C_2$ paired comparisons x_{ij} ($i < j$; $i, j, = 1 \sim 13$) at a time, then the logarithm of observation error e_{ij} has variance $\sigma^2(13) = 0.980^2$ (→ (4 · 5)). We calculate $x_{ij} = a_{ij}\, e_{ij}$ from Table 8 and the random numbers e_{ij} with above mentioned properties. $X = \{x_{ij}\}$ and its meximal eigen value and its eigen vector are shown in Table 11. These estimates almost have no reliability.

At the end of this section we note that we can use LLS estimation (→§3) for our purpose.

First we apply LLS estimation (→(3 · 8)) to Table 6 (Example 3) and have $\hat{w}_i$ as estimate of w_i ($i = 1 \sim 7$) (in Table 4). These are shown in Table 12, where w_i (in Table 6) are shown again for the comparison. (Of course $\hat{w}_i$'s are standardized as $\hat{w}_1' = 1$). We can see that $\hat{w}_i$ gives a surprisingly good approximation to w_i ($i = 1 \sim 7$).

Second appling LLS to Table 7 we have Table 13. This also gives farely good approximations. Next we have Table 14 from Table 10 and Table 15 from Table 11 by the same way as above.

Generally we can say that the LLS estimation gives a good approximation to the eigen vector estimation when the observation error is small and the number of abjects is small.

The labor of the calculation of the LLS is far easier than the eigen vector method. The former is easily done on a desk calculator of pocket size, but the latter needs at least a personal computer. So the advantege of LLS method should be highly appreciated even in the highly computerized countries like Japan, USA and etc. Never the less the author think that the deep meaning of AHP is concealed in using the eigen vector for a estimation

Table 8 13 × 13

		1	2	3	4	5	6	7	8	9	10	11	12	13
w_1 =0.1341	1	1	1.111	1.235	1.372	1.524	1.6393	1.882	2.091	2.323	2.581	2.868	3.187	3.541
w_2 =0.1207	2	0.9	1	1.111	1.235	1.372	1.521	1.693	1.882	2.091	2.323	2.581	2.868	3.187
w_3 =0.1086	3	0.81	0.9	1	1.111	1.235	1.382	1.524	1.693	1.882	2.091	2.323	2.581	2.868
w_4 =0.0977	4	0.81	0.9	1	1.111	1.235	1.372	1.524	1.693	1.882	2.091	2.323	2.581	
w_5 =0.0880	5	0.6561	0.729	0.81	0.9	1	1.111	1.235	1.372	1.524	1.693	1.882	2.091	2.323
w_6 =0.0792	6	0.5905	0.6561	0.729	0.81	0.9	1	1.111	1.235	1.372	1.524	1.693	1.882	2.091
w_7 =0.0713	7	0.5314	0.5905	0.6561	0.729	0.81	0.9	1	1.111	1.235	1.372	1.521	1.693	1.882
w_8 =0.0641	8	0.4783	0.5314	0.5905	0.6561	0.729	0.81	0.9	1	1.111	1.235	1.372	1.524	1.693
w_9 =0.0577	9	0.4305	0.4783	0.5314	0.5905	0.6561	0.729	0.81	0.9	1	1.111	1.235	1.372	1.524
w_{10}=0.0519	10	0.3874	0.4305	0.4783	0.5314	0.5905	0.6561	0.729	0.81	0.9	1	1.111	1.235	1.372
w_{11}=0.0468	11	0.3487	0.3874	0.4305	0.4783	0.5311	[illegible]	0.6561	0.729	0.81	0.9	1	1.111	1.235
w_{12}=0.0421	12	0.3138	0.3487	0.3874	0.4305	0.4783	0.5311	[illegible]	0.6561	0.729	0.81	0.9	1	1.111
w_{13}=0.0379	13	0.2824	0.3138	0.3487	0.3874	0.4305	0.4783	0.5311	0.5905	0.6561	0.729	0.81	0.9	1

Table 9 X_1, X_2 … X_{13}

X_1 =

	1	2	4	10
1	1	1.272	1.057	1.111
2	0.7862	1	0.9282	1.115
4	0.9461	1.077	1	2.648
10	0.3181	0.7067	0.3776	1

X_2 =

	2	3	5	11
2	1	0.9162	2.113	1.066
3	1.05	1	1.388	1.666
5	0.4733	0.7205	1	1.932
11	0.3262	0.6002	0.5163	1

X_3 =

	3	1	6	12
3	1	1.167	1.055	2.407
1	0.869	1	0.8383	2.667
6	0.9179	1.193	1	1.951
12	0.4155	0.375	0.5118	1

X_4 =

	4	5	7	13
4	1	1.189	1.55	1.523
5	0.841	1	0.7617	1.919
7	0.6452	1.308	1	1.846
13	0.6568	0.5132	0.5417	1

X_5 =

	5	6	8	1
5	1	1.328	1.523	1.094
6	0.753	1	1.147	0.5811
8	0.6566	0.8718	1	0.5105
1	0.9139	1.712	1.959	1

X_6 =

	6	7	9	2
6	1	1.135	1.904	0.7153
7	0.8811	1	1.15	0.5734
9	0.5252	0.8696	1	0.3708
2	1.398	1.744	2.697	1

X_7 =

	7	8	10	3
7	1	1.077	0.9667	0.5356
8	0.9285	1	0.9633	0.7553
10	1.034	1.038	1	0.722
3	1.867	1.324	1.385	1

X_8 =

	8	9	11	4
8	1	1.553	0.6957	0.9597
9	0.6439	1	1.147	0.5774
11	1.437	0.8718	1	0.4792
4	1.042	1.732	2.087	1

X_9 =

	9	10	12	5
9	1	0.9799	1.421	0.7321
10	1.021	1	1.17	0.621
12	0.7037	0.8547	1	0.5653
5	1.366	1.613	1.769	1

X_{10} =

	10	11	13	6
10	1	0.9447	1.454	0.8606
11	1.059	1	1.101	0.3994
3	3.6876	0.9082	1	0.3431
6	1.162	2.504	2.915	1

X_{11} =

	11	12	1	7
11	1	0.9362	0.2237	0.511
12	1.068	1	0.4658	0.6242
1	4.477	2.147	1	2.215
7	1.945	1.602	0.4515	1

X_{12} =

	12	13	2	8
12	1	0.9769	0.2638	0.7886
13	1.024	1	0.2467	0.5901
2	3.791	4.054	1	1.691
8	1.268	1.695	0.5914	1

X_{13} =

	13	1	3	9
13	1	0.2024	0.3372	0.8497
1	4.94	1	1.76	1.865
3	2.966	0.568	1	1.97
9	1.177	0.5362	0.5076	1

Table 9' X_1 X_2 … X_{13} (continued)

1.0000	1.2720	1.7600	1.0570	0.9139	1.7120	2.2150	1.9590	1.8650	3.1440	4.4770	2.1470	4.9400
0.7862	1.0000	0.9462	0.9282	2.1130	1.3980	1.7440	1.6910	2.6970	1.4150	3.0660	3.7910	4.0540
0.5682	1.0570	1.0000	1.1670	1.3880	1.0550	1.8670	1.3240	1.9700	1.3850	1.6660	2.4070	2.9660
0.9461	1.0770	0.8569	1.0000	1.1890	0.8383	1.5500	1.0120	1.7320	2.6480	2.0870	2.6670	1.5230
1.0940	0.4733	0.7205	0.8410	1.0000	1.3280	0.7647	1.5230	1.3660	1.6130	1.9370	1.7690	1.9490
0.5841	0.7153	0.9479	1.1930	0.7530	1.0000	1.1350	1.1470	1.9040	1.1620	2.5040	1.9540	2.9150
0.4515	0.5734	0.5356	0.6452	1.3080	0.8811	1.0000	1.0770	1.1500	0.9667	1.9450	1.6020	1.8460
0.5105	0.5914	0.7553	0.9597	0.6566	0.8718	0.9285	1.0000	1.5530	0.9633	0.6957	1.2680	1.6950
0.5362	0.3708	0.5076	0.5774	0.7321	0.5252	0.8696	0.6439	1.0000	0.9799	1.1470	1.4210	1.1751
0.3181	0.7067	0.7220	0.3776	0.6200	0.8606	1.0340	1.0380	1.0210	1.0000	0.9447	1.1700	1.4540
0.2237	0.3262	0.6002	0.4792	0.5163	0.3994	0.5141	1.4370	0.8718	1.0590	1.0000	0.9362	1.1010
0.4658	0.2638	0.4155	0.3750	0.5653	0.5118	0.6242	0.7886	0.7037	0.8547	1.0680	1.0000	0.9769
0.2024	0.2467	0.3372	0.6568	0.5132	0.3431	0.5417	0.5901	0.8497	0.6876	0.9082	1.0240	1.0000

eigen value 13.343170

vector

1.00000	0.88277	0.71639	0.70760	0.60039	0.62915	0.49856	[illegible]	0.38128	0.41178	0.3356	0.31258	0.27811
w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	w_{10}	w_{11}	w_{12}	w_{13}

Table 11 13×13 $X = [x_{ij}]$ (usual AHP)

1	2	3	4	5	6	7	8	9	10	11	12	13
1	1.774	1.071	2.175	1.919	0.5758	1.045	12.48	16.16	3.908	0.8363	1.902	3.884
0.5636	1	0.8771	3.227	1.781	0.4639	0.2651	1.411	1.370	0.9083	1.924	8.971	3.072
0.9338	1.140	1	6.522	1.000	2.094	0.4587	4.016	3.730	1.012	3.680	0.6331	2.824
0.4599	0.3099	0.1533	1	7.611	0.9521	1.959	0.1990	0.2546	0.8926	2.651	0.5842	3.456
0.5212	0.5615	0.999	0.1314	1	1.357	1.393	1.208	0.8352	1.392	8.979	6.493	0.7722
1.737	2.156	0.4776	1.050	0.7372	1	0.6666	2.710	8.762	10.76	0.6958	1.390	3.136
0.9574	3.772	2.180	0.5104	0.7178	1.500	1	1.085	0.7848	2.004	14.61	6.686	3.485
0.08013	0.7087	0.2490	5.024	0.8277	0.3691	0.921	1	1.584	0.5654	6.730	1.110	7.981
0.06187	0.7300	0.2681	3.927	1.197	0.1141	1.274	0.6312	1	1.145	1.319	0.5351	2.335
0.2559	1.101	0.9879	1.120	0.7182	0.09297	0.4989	1.869	0.8730	1	2.956	0.8129	1.162
1.196	0.5197	0.2718	0.3772	0.114	1.437	0.06842	0.1186	0.7115	0.3383	1	0.8814	1.807
0.5258	0.1115	1.579	1.712	0.1540	0.7196	0.1496	0.9006	1.869	1.230	1.135	1	0.05509
0.2574	0.3255	0.3541	0.2894	1.295	0.2911	0.2869	0.1253	0.1283	0.8563	0.5535	18.1	1

eigen value = 19.80188

vector =

$\hat{w}_1$	$\hat{w}_2$	$\hat{w}_3$	$\hat{w}_4$	$\hat{w}_5$	$\hat{w}_6$	$\hat{w}_7$	$\hat{w}_8$	$\hat{w}_9$	$\hat{w}_{10}$	$\hat{w}_{11}$	$\hat{w}_{12}$	$\hat{w}_{13}$
1	0.4795	0.6115	0.1551	0.4517	0.0925	0.7016	0.5000	0.3069	0.2703	0.2202	0.2179	0.3613

Table 12
(Decomposition)

i	$\hat{w}_i$ (eigen vector)	$\tilde{w}_i$ (LLS)
1	1	1
2	0.6067	0.6061
3	0.0498	0.0498
4	0.1222	0.1218
5	0.1136	0.1137
6	0.2916	0.2904
7	0.2131	0.2134

Table 13
(AHP)

i	$\hat{w}_i$ (eigen vector)	$\tilde{w}_i$ (LLS)
1	1	1
2	0.7679	0.78601
3	0.0793	0.0755
4	0.1303	0.1273
5	0.1460	0.1385
6	0.4139	0.4067
7	0.2201	0.2224

Table 14
(Decomposition)

i	$\hat{w}_i$ (eigen vector)	$\tilde{w}_i$ (LLS)
1	1	1
2	0.8028	0.8066
3	0.7164	0.7354
4	0.7076	0.7126
5	0.6064	0.6093
6	0.6292	0.6439
7	0.4986	0.5102
8	0.4663	0.4723
9	0.3813	0.3926
10	0.4148	0.4206
11	0.3356	0.3354
12	0.3126	0.3209
13	0.2781	0.2833

Table 15
(AHP)

i	$\hat{w}_i$ (eigen vector)	$\tilde{w}_i$ (LLS)
1	1	1
2	0.4795	0.4002
3	0.6115	0.6037
4	0.1551	0.7080
5	0.4517	0.5109
6	0.0925	0.7705
7	0.7016	0.8428
8	0.5000	0.1611
9	0.3069	0.3210
10	0.2703	0.3635
11	0.2202	0.2123
12	0.2179	0.2196
13	0.3613	0.2511

Conclutions

For the statistical model stated in (3・1) (3・2) our BIBD methog gives better results than the usual AHP method. Further the LLS method (3・8)) gives very good approximation to the eigen value method.

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