Observations on Multiplicative Composition in the Analytic Hierarchy Process

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1. Introduction

From the early years of development of the Analytic Hierarchy Process (AHP) I examined a number of alternatives for hierarchic composition that produced ratio scales. Ratio scales were always in my thinking because of my strong interest in physics. There one multiplies different ratio scales before and after raising them to powers. But even in mathematics, length, area and volume involve different powers of a metric. In my first book on the AHP I pointed out what is commonly known, that the product and quotient of different ratio scales is a ratio scale but not the sum or difference. The sum of elements from the same ratio scale is in fact a ratio scale. I also gave a theorem on the powers of a ratio scale which as products of ratio scales define a ratio scale. Later I worked out examples using multiplicative composition where to obtain the composite weight of an alternative, the weight of that alternative is raised to the power of its criterion and the product is taken over the criteria. In all the examples the outcome was close to that obtained by additive composition which is simply multiplying the weight of an alternative by that of its corresponding criterion and adding over the criteria. In this note I want to give several reasons for not using the multiplicative approach for hierarchic composition.

2. Problems With Multiplicative Composition

The AHP is a methodology that derives ratio scales from people's judgments. These judgments are intrinsically inconsistent. It is a good thing that we can allow for inconsistency in judgments for then they can be incrementally modified to allow for the effect of new information. Judgments are captured through paired comparisons expressing the dominance: importance, preference, or likelihood of one element over another. If the matrix is consistent, one can derive the rank of the elements by many different ways which all lead to the same

outcome. One of them is based on multiplying the elements. However, another is based on adding the elements. When we admit inconsistency to capture numerical dominance, we need to both add and multiply the numerical values to capture the dominance of one element over another B either directly or through intermediate paths of a pair of judgments, triple judgments, and so on at infinitum. Such transitivity is obtained by raising the matrix of judgments to all order powers and combining the results which yield the principal right eigenvector of the matrix of comparisons for the ranking weights. Thus when inconsistency is allowed, the eigenvector is the unique way to obtain the rank. It is obtained by both multiplying and adding judgments. To avoid the addition operation some people have felt justified demand that the judgments always be consistent, more as wishful thinking than a valid observation. The following example shows that with inconsistency, multiplicative composition leads to different (and wrong) results for the values and ranks of the alternatives.

Consider the inconsistent matrix

	A	B	C	D	E	eigenvector (Additive) EM	(Multiplicative) LLSM
A	1	1/7	1/2	1/8	2	.0612	.0616
B	7	1	3	1	8	.3743	.3839
C	2	1/3	1	1/4	5	.1342	.1328
D	8	1	4	1	5	.3869	.3801
E	1/2	1/8	1/5	/15	1	.0434	.0416

where $\mu(n) = .045$ and $\mu(n)$ [Random $\mu(n)$] $= .040$

The rankings are

$$w_D > w_B > w_C > w_A > w_E$$

$$u_B > u_D > u_C > u_A > u_E$$

Again we note the counterintuitive outcome that if some element in an inhomogeneous set has a low priority relative to others, raising this priority to the power of its criterion (which is less than one) can make its global priority larger than its local priority. For example, if the local priority is 0.02 and the criterion priority is 0.50, the global priority is 0.14147 > 0.02. Also, if an element does not have a certain property and is directly assigned a zero value for that property, its multiplicative composition would net it a priority of zero on all the criteria.

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Our third observation on not using multiplicative composition is derived from the analysis of dependence. The supermatrix is the AHP vehicle to represent dependence among the elements of a decision. Powers of the supermatrix capture the transitivity of dominance to obtain the rank along lines similar to those outlined above for the eigenvector. Additive hierarchic composition (not multiplicative composition) is a special case of the general approach of the supermatrix. In other words, additive composition is a corollary of the supermatrix approach.

Our fourth observation concerns rank. Claims have sometimes been made that multiplicative composition always preserves rank given as a reason for its adoption. But what does one do when rank should not be preserved? Is there yet another multiplicative method to allow rank to reverse? Real life examples have been given to show that a ranking method must preserve rank in some case and also allow rank to reverse in other cases. With additive composition in the AHP, the manner in which the weights of the alternatives are processed with respect to each criterion and then weighted and summed determines whether the outcome allows for rank reversal or preservation from irrelevant alternatives. In the distributive mode they are normalized and rank reversal can occur with respect to several criteria, in the ideal mode for each criterion the weights of the alternatives are divided by the largest value among them and rank is preserved.

3. The Remainder Term - Multiplicative and Additive Give Close Results

Our fifth reason for not using multiplicative composition is based on examining the closeness of ratios obtained from both additive and multiplicative composition. Let the relative weights of the criteria be $w_1, \ldots, w_n$ with $\sum_{j=1}^{n} w_j = 1$.

Let $w_{ij}, \sum_{i=1}^{m} w_{ij} = 1, j = , \ldots, n$ be the relative weight of alternative i with respect to criterion

j. Multiplicative composition yields $\prod_{j=1}^{n} w_{ij}^{w_j}$ $i = 1, \ldots, m$ for the composite weight of alternative

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i with respect to all the criteria. Now

$$\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \cdots = (x-1) + R \quad 0 < x < 2.$$

where R is the remainder. This series converges to x in the indicated range and hence its remainder is bounded.

Theorem: $\prod\limits_{j=1}^{n} w_{ij}^{w_j} = \sum\limits_{j=1}^{n} w_j w_{ij} + R_i \quad i = 1, \ldots, m$

Recalling that the alternatives are few and homogenous so that none of them is zero (or near zero) we have

$$\log \prod\limits_{j=1}^{n} w_{ij}^{w_j} = \sum\limits_{j=1}^{n} w_j \log w_{ij} = \sum\limits_{j=1}^{n} w_j(w_{ij}-1) + Q_i = \left[\sum\limits_{j=1}^{n} w_j w_{ij} - 1 + Q_i\right] \quad i = 1, \ldots, m$$

After taking the logarithm we need to take the exponential to restore the original expression. If we use R_i to denote the remainder in our series expansion we have:

$$\exp [\quad] = 1 + \sum\limits_{j=1}^{n} w_j w_{ij} - 1 + R_i \quad i = 1, \ldots, m$$

where R is the remainder.

Thus, to a first order approximation we have:

$$\prod\limits_{j=1}^{n} w_{ij}^{w_j} = \sum\limits_{j=1}^{n} w_j w_{ij} + R_i \quad i = 1, \ldots, m \tag{1}$$

Because the first term on the right is normalized to unity when summed over i, we also normalize the left side so that

$$\sum\limits_{i=1}^{m} \prod\limits_{j=1}^{n} w_{ij}^{w_j} = 1$$

and (1) becomes

$$\prod\limits_{j=1}^{n} w_{ij}^{w_j} \Big/ \sum\limits_{i=1}^{m} \prod\limits_{j=1}^{n} w_{ij}^{w_j} = \sum\limits_{j=1}^{n} w_j w_{ij} + R_i \quad i = 1, \ldots, m \tag{2}$$

The question raised by the above representation does not necessarily have to do with the size of the remainder. The representation in (2) suggests two ideas. The first is that the right side be written as a multiplicative perturbation of the left side. Thus instead of adding a remainder, we write

$$\prod_{j=1}^{n} w_{ij}^{n} / \sum_{i=1}^{m} \prod_{j=1}^{n} w_{ij}^{w_j} = (\sum_{j=1}^{n} w_j w_{ij}) \varepsilon_i \qquad (3)$$

Where the perturbation ε_i is positive. The other idea is that ratios formed by using the multiplicative term on the left of (3) may be close to ratios formed by the additive term in (3). To test the closeness or the compatibility of all such ratios we performed a random simulation exercise of 2500 × 81 experiments to determine the closeness of the two sides of (2) normalized. We systematically increased the number of criteria from 2 to 10 and the number of alternatives from 2 to 10. For each number of criteria and of alternatives, we assigned the criteria and the alternatives random values from a uniform distribution on (0,1] that were then normalized and combined by the two methods. For each number of criteria and alternatives 2500 experiments were performed. Each time absolute differences were taken between the two sides for each alternative and the results were analyzed for the alternatives. The outcomes were then averaged across all 2500 experiments to yield the mean difference.

We then used our ratio scale metric $\frac{1}{n^2} e^T A \circ B^T e$ to evaluate the closeness of the same 2500 experiments for each number of alternatives and criteria. Here we formed a matrix A of all possible ratios of the multiplicative ratio scale. Each row of A gives the ratios of one element with respect to each other element. We did the same for the additive ratio scale to define a matrix B. We then formed a new matrix $A \circ B^T$ (where B^T is the transpose of B) each of whose entries is a ratio of corresponding entries of the first and of the second matrix respectively. If the scales are identical each entry of this matrix would be equal to one. Finally, we summed the elements of $A \circ B^T$ which corresponds to performing the operation $e^T A \circ B^T e$ where $e^T = (1, 1, ..., 1)$. Since the sum of the elements of $A \circ B^T$ is $\geq n^2$ where n is the order of the matrix we divided by n^2 [1]. In general, when two scales are close we say that they are

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compatible if $e^T \frac{1}{n^2} A \circ B^T e \leq 1.10$. The outcome of this exercise given in the table below shows that except for the case of two criteria and more than six alternatives, the two sides of (2) are close according to their mean differences and compatible according to their mean ratios. Statistical regression analysis shows that the mean difference values cannot be explained by the number of criteria and of alternatives. However, the mean metric values could be explained with greater confidence by using both these parameters.

4. Conclusions

Our experiments with the exception of a part of the two criteria case, show that multiplicative composition does not add significant information to the ratios obtained from additive composition. Because the ratio scales used on both sides of (3) give close results one may conclude that the two methods can be used interchangeably but that is not true. We have seen that if the judgments are inconsistent one must use the principal eigenvector and additive composition and not the multiplicative approach to capture the right priorities and ranks. This approach is also true of the supermatrix from which we deduce additive hierarchic composition as a corollary.

References

1. Saaty, T.L., 1994, "A Ratio Scale Metric and the Compatibility of Ratio Scales: On the Possibility of Arrow's Impossibility Theorem", Applied Mathematics Letters (to appear).

Crit	Alt	Mean Difference	Mean Metric	Crit	Alt	Mean Difference	Mean Metric
002	002	0.0379536077380818	1.0315046310424480	005	006	0.0246297232806 68	1.057444214820862
002	003	0.0315362960100017	1.0648034811101990	005	007	0.0211696680635 21	1.058788537979126
002	004	0.0263856742531106	1.0790646076202239	005	008	0.0184370335191 49	1.060430526733398
002	005	0.0210705287754 54	1.0881742223899841	005	009	0.0165510307997 47	1.063900709152222
002	006	0.0174668990007559	1.0979437828063 96	005	010	0.0147587079554 80	1.064010977745056
002	007	0.0150932176038 62	1.1019945144653 32	006	002	0.0639477521181 11	1.015824913978577
002	008	0.0130198840051 89	1.1110761165618 90	006	003	0.0478015802800 66	1.031071662902832
002	009	0.0116901751607 66	1.1103969812393 19	006	004	0.0367075577378 27	1.039645791053772
002	010	0.0104960529133 68	1.1175299882888 79	006	005	0.0289402212947 61	1.042592883110046
003	002	0.0529505908489 23	1.0279980897903 44	006	006	0.0244603175669 91	1.048782706260681
003	003	0.0433998741209 51	1.0552867650985 72	006	007	0.0210077222436 67	1.049988508224487
003	004	0.0338384918868 54	1.0662209987640 38	006	008	0.0184551142156 12	1.053045272827148
003	005	0.0269504971800 23	1.0822297334671 02	006	009	0.0161127373576 16	1.054090976715088
003	006	0.0227295774966 48	1.0830503702163 70	006	010	0.0146908052265 64	1.055384755134583
003	007	0.0193560123443 60	1.0919775962829 59	007	002	0.0624657757580 28	1.013782143592834
003	008	0.0169314723461 87	1.0913485288620 00	007	003	0.0475074872374 53	1.027516841888428
003	009	0.0150525011122 23	1.0958969593048 10	007	004	0.0357936657965 18	1.034428477287292
003	010	0.0135427853092 55	1.0978820323944 09	007	005	0.0289248945450473	1.037729978561401
004	002	0.0611888915300 37	1.0217952728271 48	007	006	0.0242527928203 34	1.043687343597412
004	003	0.0466310903429 99	1.0429803133010 86	007	007	0.0206472463905 81	1.045656681060791
004	004	0.0361922308802 60	1.0538612604141 24	007	008	0.0179959107190 37	1.044478654861450
004	005	0.0289828404784 20	1.0628663301467 90	007	009	0.0160166826099 16	1.047833561897278
004	006	0.0241173449903 73	1.0698201656341 55	007	010	0.0142264738678 93	1.047522544860840
004	007	0.0211065150797 37	1.0710703134536 74	008	002	0.0635208785533 91	1.012796401977539
004	008	0.0180572904646 40	1.0711745023727 42	008	003	0.0464232489466 67	1.023950099945068
004	009	0.0161055047065 02	1.0776531696319 58	008	004	0.0347676277160 64	1.029846310615540
004	010	0.0144811403006 32	1.0757246017456 05	008	005	0.0283383764326 57	1.035024166107178
005	002	0.0639340579509 74	1.0183923244476 32	008	006	0.0237017720937 73	1.038161039352417
005	003	0.0478792712092 40	1.0346568822860 72	008	007	0.0203126166015 86	1.039396762847900
005	004	0.0365802757442 00	1.0438024997711 18	008	008	0.0175038035959 01	1.039437413215637
005	005	0.0296375757286 891	1.0526914596557 62	008	009	0.0157023705542 09	1.042114019393921

Crit	Alt	Mean Difference	Mean Metric	Crit	Alt	Mean Difference	Mean Metric
008	010	0.0140248921908	1.0419524908065 80				
009	002	0.0613296553492 55	1.0108597278594 97				
009	003	0.0453829020261 76	1.0208669900894 17				
009	004	0.0344316810369 49	1.0273836851120 00				
009	005	0.0278042107820 51	1.0314480066299 44				
009	006	0.0231018736958 50	1.0335406064987 18				
009	007	0.0196957910478 1	1.0348823070526 12				
009	008	0.0172044225037 10	1.0366158485412 60				
009	009	0.0153156314045 19	1.0379916429519 65				
009	010	0.0137228090316 06	1.0380815267562 87				
010	002	0.0606007613241 67	1.0099923610687 26				
010	003	0.0441154092550 28	1.0188766717910 77				
010	004	0.0336349196732 04	1.0245430469512 94				
010	005	0.0267286896705 63	1.0276254415512 08				
010	006	0.0221278369426 73	1.0293846130371 09				
010	007	0.0191838890314 10	1.0316343307495 12				
010	008	0.0167098753154 28	1.0331077575683 59				
010	009	0.0148330302909 02	1.0337196585851 624				
010	010	0.0133698042482 14	1.0349931716918 95				