THE ANALYTIC HIERARCHY PROCESS (AHP) WITH BOUNDED INTERVAL INPUT

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ABSTRACT

The AHP has used the point estimation only for its input data. That is, the AHP assumes that no error exists in a set of pairwise judgements (comparisons) data as long as the desired level of logical consistency of judgements is achieved. If the original purpose of AHP is to solve complicated, unstructured, fuzzy decision problems, then its input can hardly be free of estimation errors. The interval estimation, which contains the estimation error, may be more natural for the AHP input. If the input to AHP problems are truly interval estimation, then they should be operated as such, because the incremental computational burden is marginal when compared to incremental information from the bounded interval solution. The AHP operation with interval input is proposed and a numerical example is illustrated.

INTRODUCTION

The Analytic Hierarchy Process (AHP) was introduced by Saaty (1977, 1980) as a multicriteria decision method that uses hierarchic structures to represent a decision problem and then develops priorities for the alternatives based on decision maker's (DM) judgement throughout the system. The imaginative nature of the method, which is extremely well fitted to solve complicated, intangible decision problems, has led to diverse application areas and has built a voluminous literature body. See a special issue on AHP (Harker, 1986) and consult Zahedi (1986) for a survey of the method and its applications.

One of the significant advantages of AHP over the other multiple attribute decision methods (Hwang and Yoon, 1981) is its flexible input data requirement, which is preferred by top managers (or leaders) of an organization. The input data consists of a set of pairwise comparisons of decision elements. The DM is allowed to have inconsistencies to some degree in his/her preference judgements. The AHP assumes that no error exists in the input data as long as the desired level of consistency is achieved through the consistency check. In other words, a point estimation is only used for input data of the AHP. But the interval estimation, which considers the possible inaccuracy or imprecision of the preference judgement, is more pragmatic. Saaty and Vargas (1980) indicate the possibility of interval estimation in a 1 to 9 intensity scale on some occasions. Bartoszynski and Puri (1981) question the credibility of AHP results as point estimators of uncertain outcomes and suggest interval estimation.

The purpose of this paper is to perform AHP with interval estimation as input data and to render AHP results by way of interval estimations. First, we recommend the propagation of errors technique (Pugh and Winslow, 1966) for the interval analysis. Second, the AHP with interval input is presented. Third. an example is solved to illustrate the proposed approach.

THE PROPAGATION OF ERRORS TECHNIQUE FOR INTERVAL ANALYSIS

When we have interval pairwise comparisons like $(a_{ij} \pm \Delta a_{ij})$ where Δa_{ij} is the maximum error of a_{ij}, it is not difficult to imagine that the resulting priorities (or weights) should be exposed by the bounded interval of $(w_j \pm \Delta w_j)$ too. We then need to introduce a proper interval analysis while performing an eigenvector prioritization process. The aim of interval analysis is to find the maximum possible error of the function $y = f(x_1,\ldots,x_n)$ when each variable x_j has bounded interval of $(x_j \pm \Delta x_j)$. That is to find Δy which satisfies the equation below:

$$(y - \Delta y) \le f(x_1 \pm \Delta x_1,\ldots, x_n \pm \Delta x_n) \le (y+\Delta y) \tag{1}$$

We first think about interval arithmetic (Deif, 1986). The rules of interval algebra offer a loose interval for any algebraic operation of two variables. Hence the final result after some consecutive operations (which is the case of eigenvector prioritization) usually renders a quite wide interval which may not be useful. Consequently, the option of interval arithmetic is excluded.

The other option is the propagation of errors technique proposed by Pugh and Winslow (1966). They pointed out that the purpose of the propagation of errors is to answer the question, "Given some set of numbers and their errors, what is the error in some prescribed function involving these numbers?" Since the interval (range) of a distribution is proportional to its standard deviation, they obtained the propagation of errors of a function $y = f(x_1,\ldots, x_n)$ in the statistical fashion:

$$\sigma_y^2 = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j} \sigma_{x_j}\right)^2 \tag{2}$$

where σ_{x_j} is the standard deviation of variable x_j. Accordingly, if we use σ_y and Δy interchangeably, and use σ_{x_j} and Δx_j interchangeably, the propagation of errors equation (2) is rewritten by

$$(\Delta y)^2 = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x_j} \Delta x_j\right)^2 \tag{3}$$

THE AHP WITH INTERVAL PAIRWISE COMPARISON DATA

The details of AHP are described in Saaty (1980). Here we outline the major components of the AHP and add details only for the necessary modification due to the imprecise judgements expressed by the bounded interval. Though Saaty's eigenvector prioritization technique is clearly the winner among methods for retrieving priorities from a set of pairwise comparisons, its complicated algebraic process becomes a major hindrance in implementing the propagation of errors technique. We will use the arithmatic mean after normalization, which is also suggested by Saaty (1982). While it lacks the theoretical sophistication of the eigenvector prioritization method, it provides sufficiently close results for most cases. Furthermore, we doubt the benefit of implementing the eigenvector method for imprecise information like interval data. The four operational steps of AHP are given below:

Step 1 - Setting up the decision hierarchy by breaking down the decision problem into a hierarchy of interrelated decision elements.

Step 2 - Collecting input data. Assess $n(n-1)/2$ importance ratios between decision elements. Store this information in the upper (or lower) triangle of a $(n \times n)$ matrix whose typical element a_{ij} represents the weight ratio of w_i/w_j. Fill the remaining elements of the matrix by using the reciprocal property of the matrix: $a_{ij} = 1/a_{ji}$ and $a_{jj} = 1$ for all i and j. If the DM feels uncomfortable by point estimation, one may use interval estimation $[a_{ij}^L, a_{ij}^U]$ where a_{ij}^L is the lower and a_{ij}^U is the upper bound of a_{ij}. This interval input takes the different arithmatic expression of $(a_{ij} \pm \Delta a_{ij})$ where $a_{ij} = (a_{ij}^L + a_{ij}^U)/2$ and $\Delta a_{ij} = (a_{ij}^U - a_{ij}^L)/2$.

Step 3 - Establishing priorities. The weight (or priority) for the i^{th} decision element $(w_i \pm \Delta w_i)$ from the interval input $(a_{ij} \pm \Delta a_{ij})$ is given by

$$w_i = \frac{1}{n} \sum_{j=1}^{n} r_{ij}, \qquad i=1,\ldots,n \tag{4}$$

$$\text{where } r_{ij} = a_{ij} / \sum_{k=1}^{n} a_{kj}, \qquad i=1,\ldots,n;\ j=1,\ldots,n$$

The error term of w_i is obtained by using the propagation of error equation (3):

$$\Delta w_i^2 = \frac{1}{n^2} \sum_{j=1}^{n} \Delta r_{ij}^2, \qquad i=1,\ldots,n \tag{5}$$

$$\text{where } \Delta r_{ij}^2 = \frac{1}{\left(\sum_{k=1}^{n} a_{kj}\right)^4} \left[\Delta a_{ij}^2 \left(\sum_{\substack{k=1 \\ k\neq i}}^{n} a_{kj} \right)^2 + a_{ij}^2 \left(\sum_{\substack{k=1 \\ k\neq i}}^{n} \Delta a_{kj}^2 \right) \right],$$

$$i=1,\ldots,n,\ j=1,\ldots,n.$$

Finally, the λ_{max} for the consistency check for the input is approximated by

$$\lambda_{max} = \frac{1}{n} \sum_{i=1}^{n} c_i / w_i \qquad (6)$$

where $c_i = \sum_{j=1}^{n} a_{ij} w_j$, $\quad i=1,\ldots,n$

The error term of λ_{max} by way of equation (3) is given as

$$\Delta\lambda_{max}^2 = \frac{1}{n^2} \sum_{i=1}^{n} \left[\frac{1}{w_i^2} \Delta c_i^2 + \frac{c_i^2}{w_i^4} \Delta w_i^2 \right] \qquad (7)$$

where $\Delta c_i^2 = \sum_{j=1}^{n} \left[a_{ij}^2 \Delta w_j^2 + w_j^2 \Delta a_{ij}^2 \right]$, $\quad i=1,\ldots,n$

The range of λ_{max} is $(\lambda_{max} \pm \Delta\lambda_{max})$, but the lower bound of λ_{max} should not be less than n.

Step 4 - Aggregating weight across hierarchy. If the immediately above hierarchy level has m decision elements and the current level has n elements, then the resulting priorities can be contained in a (m x n) priority matrix. Let B_k be the priority matrix of the kth level, k=1,...,h. Then the composite priority vector (i.e., the desirability of each alternative) is obtained by

$$W = B_h B_{h-1} \cdots B_1 \qquad (8)$$

where B_1 is the unit matrix.

Let a_{ij} and b_{ij} be the typical elements of two adjacent priority matrices. The multiplication of two matrices $||a_{ij}||$ and $||b_{ij}||$ is defined by

$$||c_{ij}|| = \sum_{k=1}^{n} a_{ik} b_{kj}, \ \forall_i \text{ and } \forall_j \qquad (9)$$

Then the propagation of errors of c_{ij} due to error terms, Δa_{ij} and Δb_{ij} is given by

$$\Delta c_{ij}^2 = \sum_{k=1}^{n} \left(a_{ik}^2 \; \Delta b_{kj}^2 + \Delta a_{ik}^2 \; b_{kj}^2 \right) \tag{10}$$

This computational step is applied h times to obtain W with error terms.

A SIMPLE ILLUSTRATION

To see how the ordinary AHP process can be extended to the bounded interval input, consider the problem of a woman who has recently earned her Ph.D. and is being interviewed for three jobs. Which one should she choose? Figure 1 shows how she structured the elements of the problem and arranged them in a hierarchy. Level 1, the focus, is overall job satisfaction; level 2 comprises the criteria that contribute to job satisfaction; and level 3 consists of the three job possibilities (Saaty, 1982).

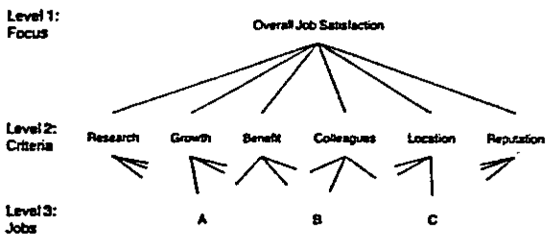

Figure 1 Hierarchy for choosing among three job offers.

The woman compared the level 2 criteria in pairs with respect to job satisfaction and judged the relative importance of each criterion. She is advised to follow Saaty's 9-point intensity scale of importance between criteria:

If criteria X_j and X_k are equally preferred, then X_j/X_k is 1.

If X_j is moderately preferred to X_k, then X_j/X_k is 3.

If X_j is strongly preferred to X_k, then X_j/X_k is 5.

If X_j is very strongly preferred to X_k, then X_j/X_k is 7.

If X_j is extremely preferred to X_k, then X_j/X_k is 9.

Intermediate values 2, 4, 6, and 8 may be used when compromise is needed.

Table 1 shows her pairwise comparison matrix of the criteria with respect to the focus. Some elements of the matrix are expressed by a range, which reflects the DM's imprecise judgement. For instance, benefit is strongly or more (but less than very strongly) preferred to colleagues, etc. The last column of Table 1 shows the priority vector with its error terms obtained by equations (4) and (5). Next she developed six matrices for comparing the three jobs with respect to each criterion. Her pairwise judgements and the vector of priorities are given in Table 2.

Table 1 Overall satisfaction with the job.

Overall Job Satisfaction	Research	Growth	Benefits	Colleagues	Location	Reputation	Vector of Priorities
Research	1	1	1	[4, 5]	1	1/2	.1624 ± .0035
Growth	1	1	1	[4, 5]	1	1/2	.1624 ± .0035
Benefits	1	1	1	[5, 6]	[3, 4]	1/2	.2163 ± .0066
Colleagues	[1/5, 1/4]	[1/5, 1/4]	[1/6, 1/5]	1	1/2	[1/5, 1/4]	.0461 ± .0012
Location	1	1	[1/4, 1/3]	2	1	1/2	.1219 ± .0017
Reputation	2	2	2	[4, 5]	2	1	.2908 ± .0036

$\lambda_{max} = 6.2376 \pm .0953$

Table 2. Six matrices for comparing three jobs.

Research	A	B	C	Vector of Priorities
A	1	[1/4, 1/3]	[1/2, 1]	.1713 ± .0201
B	[3, 4]	1	[2, 3]	.5856 ± .0301
C	[1, 2]	[1/3, 1/2]	1	.2431 ± .0245

$\lambda_{max} = 3.0613 \pm .2806$

Growth	A	B	C	Vector of Priorities
A	1	[1/4, 1/3]	[1/6, 1/5]	.0972 ± .0065
B	[3, 4]	1	[1/3, 1/2]	.2914 ± .0171
C	[5, 6]	[2, 3]	1	.6114 ± .0167

$\lambda_{max} = 3.0452 \pm .1648$

Benefits	A	B	C	Vector of Priorities
A	1	3	1/3	.2468 ± .0046
B	1/3	1	[1/6, 1/7]	.0920 ± .0028
C	3	[6, 7]	1	.6611 ± .0063

$\lambda_{max} = 3.0141 \pm .0566$

Colleagues	A	B	C	Vector of Priorities
A	1	1/3	[5, 6]	.2784 ± .0089
B	3	1	[8, 9]	.6576 ± .0132
C	[1/6, 1/5]	[1/9, 1/8]	1	.0640 ± .0038

$\lambda_{max} = 3.0538 \pm .0991$

Location	A	B	C	Vector of Priorities
A	1	1	7	.4667 ± 0.
B	1	1	7	.4667 ± 0.
C	1/7	1/7	1	.0667 ± 0.

$\lambda_{max} = 3. \pm 0.$

Reputation	A	B	C	Vector of Priorities
A	1	[6, 7]	[8, 9]	.7866 ± .0062
B	[1/7, 1/6]	1	1	.1115 ± .0039
C	[1/9, 1/8]	1	1	.1019 ± .0032

$\lambda_{max} = 3.0116 \pm .0721$

The overall priorities for the jobs are obtained by multiplying each corresponding priorities which are shown in Tables 1 and 2. The priorities expressed by a range are A = [.3811, .4097], B = [.2662, .2976], and C = [.3054, .3396]. She must choose job A because the lower bound of A priority is greater than the upper bound of C.

CONCLUSION

The AHP is a powerful tool in solving complex, unstructured decision problems. Its input has been limited to point estimation which is free from estimation error. If the input requirement should be relaxed to an interval estimation, its users (particularly busy top managers/leaders) find AHP more manageable. We perform AHP with interval input by employing the propagation of errors technique, and show the incremental computational effort is worth taking.

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