THE EXPONENTIAL SCALE METHOD IN AHP

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ABSTRACT

In AHP method, each element of comparative judgement matrix should be the ratio of weights, the given "1-9 scale method" just being the difference between weights. To solve the problem, an exponential scale method with "equidistant grading and equiratio valuation", is put forward in this paper. It does not only accord with the AHP ordering principle, but eliminate the inconsistency of the comparative judgement matrix which results from "1-9" scale. So as to make the consistence of judgement matrix equivalent to the consistence of judgement thinking; the ordering weights have definite sense; and also to compute and to adjust them more simply and conveniently in some way.

INTRODUCTION

Why could unquantified things be analysed quantitatively in AHP and subjective sensation be described quantitatively, the key is that the human subjective sensation be quantified with a certain scale. The quantitative results are the basis and foundation of decision-making. It is very important that quantified method is scientific.

Suppose we have n factors c_i (i=1,2,...,n), according to the principle of AHP, judgement values c_{ij} is the ratio of importance of c_i to c_j. But "1-9" scale gives out differences between c_i and c_j. On the one hand, it causes inaccuracy for the computed results of relative weight, on the other hand it makes the consistency of judgement matrix not equivalent to consistency of judgement thinking. The article derives the "Exponential Scale Method" from Weber-Fechner's Law in an attempt to overcome the above shortcomings.

WEBER-FECHNER'S LAW

B.Weber, a German physiologist, tested and verified with psychological physic experiments that any one's sensation for differences doesn't depend on his absolute deviation, but depends on his relative deviation or ratio. G.T.Fechner, a German physicist further reasoned out the function relation between subjective sensation and objective provoking:

$$S=K.LOGR \qquad (1)$$

where, S is subjective sensation, R is objective provoking, K is Weber's constant. The formula (1) shows that objective differences increase ratio-equal, the subjective sensation increase different-equal. The law is suitable for middle provoking.

THE EXPONENTIAL SCALE METHOD

Every factor in each hierarchy in AHP model is a certain systemetic attribution or character. The criteria of the comparised judgement is rightly carried out with a certain attribution. The subjective reflection to individual attribution of a system is the human's sensation. It is the human's "different sensation" that the comparative judgement matrix decribes. For the reason of differences of compared objects in AHP lies in the middle field, Weber-Fechner's law could be used here. According to this law, we may set up a new scale method with equidistant grading and equiratio valuation, called "exponential scale method".

First of all, let c_{ij} express intensity of importance of c_i to c_j, and divide c_{ij} into a series of grades equidistantly. Such as "Equal importance", "Moderate importance of one over another", "Essential or strong importance", "Demonstrated importance", "Absolute importance", ... ect. expressed individually as

$$c_{ij} = 0,1,2,... \qquad (i,j=1,2,...,n)$$

c_{ij} is called sensible judgement of c_i to c_j. Hence, $c_{ij}=l$ ($l=0,1,2,...$) expresses that c_i is higher l grade than c_j. Moreover, $c_{ij} = -l$ ($l=0,1,2,...$) expresses that c_i is lower l grade than c_j.

Secondly, according to Weber-Fechner's law that while subjective sensation increases different equal, the objective differences increase ratio-equal. We may let ratio between two grades of neighbour objective importance be a ($a>1$). Thus, the objective importance ratio of c_i to c_j is shown as follows.

$$\frac{w_i}{w_j} = a^{c_{ij}} \qquad (2)$$

$a^{c_{ij}}$ is called objective different judgement of c_i to c_j, in which w_i and w_j express intensity of objective importance of c_i and c_j respectively.

Third, to set up objective different judgement matrix

$$A = (a^{c_{ij}})_{n\times n} \qquad i,j=1,2,...,n.$$

It corresponds to comparative judgement matrix which is set up by T.Satty and is still a reciprocal matrix.

It is thought that the compared factors in the field of our considered attribution are nearer, therefore, quantitative discrimination is of significance, so as to have certain precision. The value of a is able to select a proper number larger than 1 in compliance with the difference of neighbour two grades. The satisfied ordering weights should be obtained by

adjusting the value of a. The c_{ij} is permited to take an arbitary real number.

DEFINITION OF JUDGEMENT THINKING CONSISTENCE

Under the condition of equidistant grading, the thinking consistency could be described quantitatively, while sensible judgement satifies

$$c_{ij}+c_{jk}=c_{ik}^{\circ},\ i,j,k=1,2,\ \ldots,n \qquad (3)$$

the judgement thinking is called complete consistence. Thus, the consistency of judgement thinking is equivalent to consistency of objective difference jugement matrix in the exponential scale. In fact, the

$$c_{ij}+c_{jk}=c_{ik}$$

is equivalent to the formula

$$a^{c_{ij}} \cdot a^{c_{jk}} = a^{c_{ik}} \qquad i,j,k=1,2,\ldots,n \qquad (4)$$

which is a definition about complete consistency of matrix that is given out by Prof.T.Saaty.

COMPUTATION

In use, we only need to fill up sensible judgement matrix. Its characters are: (1) on the main diagonal the entries are zero, (2) the symbols of two symmetrical components are reversed, such as

	c_1	c_2	c_3	c_4	Σ
c_1	0	2	4	1	7
c_2	-2	0	2	-1	-1
c_3	-4	-2	0	-2	-8
c_4	-1	1	2	0	2

(5)

when computed it may be replaced by objective difference judgement matrix (6).

	c_1	c_2	c_3	c_4
c_1	1	a^2	a^4	a
c_2	a^{-2}	1	a^2	a^{-1}
c_3	a^{-4}	a^{-2}	1	a^{-2}
c_4	a^{-1}	a	a^2	1

(6)

Once the value of a is determined, the $\lambda_{max.}$ and the corresponding eigenvector of matrix (6) may be computed.

Using exponential scale method it gets very simple for compu-

ting eigenvector of matrix (6) by some ways (such as LLSM). From the row sum of the sensible judgement matrix (5) we may write down eigenvector with parameter a, directly,

$$\overline{W} = (a^{\frac{7}{4}},\ a^{-\frac{1}{4}},\ a^{-\frac{5}{4}},\ a^{\frac{3}{4}})^T$$
$$= (a^{\frac{7}{4}},\ a^{-\frac{1}{4}},\ a^{-2},\ a^{\frac{1}{2}})^T.$$

After determining the value of a and normalized, we can get the required eigenvector soon. Another prominent advantage is that from eigenvector $\overline{W}$ with parameter a we may get satisfied ordering weights by adjusting the value of a flexiblely.

For example, as to the sight judgement matrix about "chair brightness" given by Prof.T.Saaty, when using exponential scale method, we may obtain sensible judgement matrix and objective difference judgement matrix respectively, which is as follows:

	c_1	c_2	c_3	c_4
c_1	0	3	5	6
c_2	-3	0	2	3
c_3	-5	-2	0	1
c_4	-6	-3	-1	0

(7)

	c_1	c_2	c_3	c_4
c_1	1	a^3	a^5	a^6
c_2	a^{-3}	1	a^2	a^3
c_3	a^{-5}	a^{-2}	1	a
c_4	a^{-6}	a^{-3}	a^{-1}	1

(8)

On account of the difference between arbitary two columns of matrix (8) is only a ratio factor. Therefore, we may select arbitary column as eigenvector, in particular, column 4 is

$$\overline{W} = (a^6, a^3, a^1, a^0)^T$$

when a = 1.2

$$\overline{W} = (2.986,\ 1.728,\ 1.2,\ 1)^T$$

After normalized $\overline{W}$, we can obtain required eigenvector.

$$W = (0.4318,\ 0.2499,\ 0.1736,\ 0.1446)^T$$

if $a = \sqrt{2}$, then

$$\overline{W} = (8,\ 2\sqrt{2},\ \sqrt{2},\ 1)^T$$

After normalized $\overline{W}$, we have

$$W = (0.604,\ 0.214,\ 0.107,\ 0.076)^T$$

The later is $2^{\frac{n}{5}}$ - scale, one of the two better scales used by Prof.T.Saaty at first.

CONCLUSION

The advantages using the exponential scale are
1). To point out the principle of equidistant grading for sensation and definition about consistence of thinking, to determine sensible judgement conveniently, to keep thinking consistence easily.
2). To simplify computation with some appropriate ways.
3). To adjust the ordering weights conveniently by parameter.
4). The objective difference judgement $a^{c_{ij}}$ represents the objective important ratio of c_i to c_j. It accords with the ordering principle in AHP. Hence, it causes the ordered weights to have definite sense, which expresses the relative importance of the factors. While a is determined, especially, the researched object being measurable physical quantity, the ordered weights will be of reality and conform with "relative weights" of actual physical quantity, while a and c_{ij} tally with actuality.
5). The survey level satisfies what computation demands. As the exponential scale method has a zero point with actual meaning and has an equal unit (grade), it is a "measured rule of proportion". Its measured data are ordered, equidistant and additive. It is helpful to ensure the computed results and to provide the possibility about ordered weights tallying with actuallity.

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