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Relative Priority Shifts and Rank Reversals in AHP

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The paper begins with a discussion of the legitimacy of relative priority shifts and rank reversals. This is followed by an analysis of why rank reversals occur in AHP. The paper ends with a brief discussion of modifications to AHP in which priority shifts and rank reversals do not occur.

Over the years, AHP has been criticized and rejected by some decision theorists on the basis of rank reversals, and what they consider to be a lack of specificity in the evaluation of criteria importance. Others, no doubt including the large majority of people at this conference, point to AHP's simplicity and intuitive appeal, and are untroubled by rank reversals. I would place myself in a small camp who believe that AHP can and should be modified to address the objections of its critics, while still maintaining those positive elements which have contributed to its popularity.

It has been argued elsewhere [10] that conventional AHP fails on problems with known answers, and these arguments will not be repeated here. I will argue here against the position taken by some that rank reversals and priority shifts in AHP are legitimate, and will discuss the mechanism which causes them to occur. I will close with a demonstration of two modifications to AHP in which rank reversals and priority shifts do not occur.

In the discussion which follows I will assume that we have perfect consistency in paired comparisons, thus eliminating inconsistency as a possible cause of any anomolous results. I assume that we are all familiar with the fact that rank reversals may occur if an alternative is added or deleted to the choice set in the conventional AHP procedure. I am referring to reversals which occur under the following conditions: the problem involves alternatives evaluated using relative measurement, there is more than one criterion involved, and the paired comparison values of the local priorities of the existing alternatives remain unchanged. Belton and Gear presented the first example of such a reversal in the form of a problem in which a copy of the preferred alternative was added to the choice set [1]. But copies are not a necessary condition for such reversals to occur.

Consider the following example. Alternatives A and B are compared on criteria C_1 and C_2, where the two criteria are considered equally important. The local priorities and synthesized priorities are shown below.

I would like to thank Eng Choo and William Wedley for their valuable comments and suggestions on an earlier draft. This, of course, does not imply that they agree with everything in this version of the paper, or are responsible for any errors which remain.

	C_1	C_2
	1/2	1/2
A	1/4	3/4
B	3/4	1/4

$$W_A = 1/4x1/2 + 3/4x1/2 = 0.5 = W_B$$

Now suppose that a third alternative C is added to the choice set, and local and synthesized priorities become:

	C_1	C_2	
	1/2	1/2	
A	1/6	3/5	$W_A = 1/2x1/6 + 1/2x3/5 = 23/60$
B	3/6	1/5	$W_B = 1/2x3/6 + 1/2x1/5 = 21/60$
C	2/6	1/5	$W_C = 1/2x2/6 + 1/2x1/5 = 16/60$

With only A and B in the choice set, we are indifferent between them. We add C to the choice set, where C is a clearly inferior alternative dominated by B. Although we have changed neither the relative ratings of A and B under the individual criteria nor the criteria weights, we now find that A is preferred to B. Indeed, in this example the only instance in which we would not observe a shift in the relative preference of A and B would be if the local priority of C would be identical under the two criteria. Finally, we observe that the addition of an alternative to the choice set will generally cause a shift in the synthesized priorities of the existing alternatives, even though actual rank reversal may not occur.

The question is whether this phenomenon reflects a difficulty in the AHP procedure or, alternatively, a natural shift which AHP uncovers. I take the former view, although I also feel that my colleagues and I have shown that the difficulty is easily overcome.

On the legitimacy of relative priority shifts and rank reversals

A number of arguments have been proposed to justify rank reversals in AHP [2,3,5,6,7] I will discuss those arguments which I am consider to be most important, although I make no claim to have exhausted the list. But first I would like to delineate the circumstances under which a rank reversal would, in my view, be justified. It may be the case that the addition of an alternative adds information which leads the decision-maker to change paired comparison values of alternatives on the criteria, paired comparison values of the criteria, or suggests a new relevant criterion. A rank reversal may appear because of this new information. It is a characteristic of such reversals that they do not revert to the original ranking if the new alternative is withdrawn. In this sense, the new alternative may be thought of as a carrier of the information, but not necessary to the reversal in any other sense. In contrast, I would characterize an unjustified rank reversal as one which occurs in spite of the fact that the paired comparison values of existing alternatives or criteria are unchanged, and no new criterion is suggested by the new alternative.

As an example of a justified rank reversal, a person may be considering purchasing a family sedan, with reliability, price and comfort as criteria. The addition of BMW to the choice set may remind him to also include manouverability as a criterion. If one of the existing alternatives in the choice set is more manouverable than the others it may well move up in rank. This move is likely to remain even though the BMW is dropped from the set.

Arguments for rank reversal

(1) Consider the example of the traveller in Luce and Raiffa [4]. Perusing the menu at an unfamiliar restaurant, he is torn between steak and salmon. If he had confidence in the chef he would choose salmon, but he actually chooses the steak on the premise that it would be less subject to damage in the hands of an incompetent cook. Upon being told by the waiter that frogs legs, though not on the menu, are available as a special, the diner concludes that the chef must be capable indeed, and changes his order to salmon.

This example has been used to justify rank reversals produced by the addition of a new alternative. I too would agree that this reversal is justified. In this example, rank reversal occurs because the addition of a third alternative adds information which changes the paired comparison values of the first two alternatives on the criteria (actually, a single criterion is involved.). Indeed, if the waiter came back from the kitchen to announce that it was fortunate that our traveller had not ordered the frogs legs inasmuch as the last of the legs had been ordered at another table, the preference of the traveller would not revert back to the steak.

(2) The argument of relative scarcity has been employed to justify rank reversals. Recall that the Belton and Gear rank reversal was produced by the addition of a copy of the highest ranking alternative to the choice set. But this implies that the added alternative is now relatively plentiful. Since scarcity adds value to goods, it is not surprising that a rank reversal occurs.

I believe there are a number of flaws in this argument. First, if scarcity is a relevant criterion it should be added to the list of criteria, with the alternatives directly compared with respect to their relative scarcity. One should not require the addition of another alternative to provide this information. Second, in economics a more plentiful availability of some good is reflected in a shift in the supply curve, with no necessary shift in consumer preferences. Unless we are dealing with a so called "inferior good", increased supply will lead to decreased price. With price as one of the relevant criteria, the consumer is likely to consume more of the more plentiful good, not less.

Finally, it is not entirely clear what we mean by a copy. In evaluating summer activities on the basis of two criteria, enjoyment and cardiovascular benefit, I may find that tennis is superior overall to golf. But suppose I am now introduced to hiking and add it to the choice set. If I rate hiking and tennis exactly the same on each of the two criteria, my AHP result may now be that golf has the highest synthesized priority. Thus I spend my summer playing golf, in spite of the fact that pairwise, I prefer either tennis and hiking to golf.

Perhaps I should be prohibited from adding hiking to the choice set inasmuch as it is considered a copy of tennis. But it is clear that hiking and tennis are distinct activities, and not copies. I may be unique in rating them to be equal on the criteria. Furthermore, if one is untroubled by such reversals, why should the addition of an alternative be prohibited?

(3) The work of many researchers, most notably Kahneman and Tversky on Prospect Theory [11] demonstrates that rank reversals are part of life. In particular, if a lottery is framed in terms of savings, K&T have shown that people tend to adopt a risk averse strategy; but if the same lottery is framed in terms of losses, people tend to adopt a risk taking strategy.

K&T make it clear that Prospect Theory is descriptive, not normative. In an interview for the DISCOVERING PSYCHOLOGY series which appeared on PBS in 1990, Tversky motivated their work on the basis of discovering how human intuition can systematically lead one astray; and how decision makers can become better aware of the pitfalls to which their intuition can take them. If a problem framed one way leads to a selection of alternative A, but framed another way leads to a selection of alternative B, their message is not "reversals are acceptable", it is "beware of the framing effect".

An argument against rank reversal in AHP

Finally, I would like to repeat an argument against rank reversals in AHP which appeared in [9]. Absolute measurement and relative measurement are both permissable in AHP. We might employ the former for repetitive decisions, in cases where we wish to maintain standards for decisions made over time, or simply because the large number of alternatives to be considered makes it desirable to reduce the magnitude of the problem by establishing standards. Thus absolute measurement may be useful in such decisions as MBA admissions, although relative measurement can still be employed. Rank reversals cannot take place when absolute measurement is used, but can with relative measurement.

Suppose an admissions committee makes a tentative ranking of students employing absolute measurement. When a new batch of applications arrive the next day, this will have no effect on the ranking of the first day's applications. But if that same committee had been examining the same applications on the same criteria, except that relative measurement was used, the second day's applications might well have changed the ranking of the first day's applications. Thus we have the unsatisfactory result that for some students whether they are accepted or rejected depends upon which technique is employed, yet both techniques are deemed acceptable within AHP.

Why do rank reversals occur in AHP?

In AHP one takes a linear composite of a set of ratio scales of local priorities in order to create a ratio scale of synthesized global priorities. It is shown below that a linear composite of a set of ratio scales is not necessarily a ratio scale. More precisely, ratio scale properties of a composite scale can be destroyed by seemingly permissable transformations of the individual scales from which it is composed.

Let w_j, $j = 1,2...m$ represent criteria weights which sum to one, the unnormalized local priorities x_{ij} represent the ratio scaled measure of the value of alternative i on criterion j, and y_i represents the ratio scaled synthesized value of alternative i. If we were forming a composite scale from unnormalized local priorities we would have:

$$y_i = w_1x_{i1}+w_2x_{i2}+...+w_mx_{im} \tag{1}$$

Comparing the ratio of the global priority of alternative g to that of alternative h we get:

$$R_{gh} = \frac{y_g}{y_h} = \frac{w_1x_{g1}+w_2x_{g2}+...+w_mx_{gm}}{w_1x_{h1}+w_2x_{h2}+...+w_mx_{hm}}, \tag{2}$$

The vector of local priorities with respect to criterion j, $X_j=(x_{1j},x_{2j}...,x_{nj})$, being ratio scaled, is unique up to a proportional transformation. Thus we can multiply local priority vector X_j by a scalar k_j which normalizes local priorities to sum to one without affecting the ratio scale property of X_j. The effect on the composite ratio scale of these separate transformations is given by:

$$R^{(1)}_{gh} = \frac{y^{(1)}_g}{y^{(1)}_h} = \frac{w_1k_1x_{g1}+w_2k_2x_{g2}+...+w_mk_mx_{gm}}{w_1k_1x_{h1}+w_2k_2x_{h2}+...+w_mk_mx_{hm}} \tag{3}$$

Equality of R_{gh} and $R^{(1)}_{gh}$ is only ensured in the special case of all the values of k_j, j= 1,2 ... m, being of equal magnitude.

The values of the normalization scalars are dependent on the particular set of alternatives in the choice set, and they will generally not be of the equal magnitude. Furthermore, adding a new alternative to the choice set will require a new set of normalization scalars k'_j. The ratio of the priorities of alternatives g and h becomes:

$$R^{(2)}_{gh} = \frac{y^{(2)}_g}{y^{(2)}_h} = \frac{w_1k'_1x_{g1}+w_2k'_2x_{g2}+...+w_mk'_mx_{gm}}{w_1k'_1x_{h1}+w_2k'_2x_{h2}+...+w_mk'_mx_{hm}} \tag{4}$$

The necessary condition for the ratios defined in (3) and in (4) to be equal is that all pairs of normalization constants retain their relative magnitudes; i.e., $k_p:k_q=k'_p:k'_q$ for all $1\le p,q\le m$.

Thus we have a process whereby unnormalized local priorities can lead to one solution, normalized to another, and normalized local priorities of an augmented choice set to a third. In [10] it is shown that none of these are likely to be correct.

Referring back to the example at the beginning of this paper, with only A and B in the choice set, the solution eigenvectors and normalization constants would be:

$$X_1 = \begin{bmatrix}1\\3\end{bmatrix},\ k_1 = 1/4;\ X_2 = \begin{bmatrix}3\\1\end{bmatrix},\ k_2 = 1/4;\ \text{and } k_1:k_2=1.$$

With C added to the choice set, the solution eigenvectors and normalization constants become:

$$X_1 = \begin{bmatrix}1\\3\\2\end{bmatrix},\ k'_1 = 1/6;\ X_2 = \begin{bmatrix}3\\1\\1\end{bmatrix},\ k'_2 = 1/5;\ \text{and } k'_1:k'_2=5/6.$$

The change in the ratios of the normalization constants is the cause of the shift in relative priorities of alternatives A and B.

A quote from Professor Saaty in [7] may help to clarify our differences. He wrote:

> ...the product of ratio scales is a ratio scale, and the sum of ratio scale numbers from the same underlying scale, as is the case with priorities derived in the AHP, is a number that belongs to that ratio scale.

I would claim that local priorities are not on the same ratio scale, and weighting them by fixed criteria importances does not put them on the same scale.

Consider the manner in which local priorities are synthesized. We begin with unnormalized local priorities. At this stage, we could conceivably elicit criteria weights from the decision-maker from which a composite ratio scale would be formed, and this scale would reflect that decision-maker's preferences. The necessary effect of the criteria weights must be to transform local priorities to common units. Or we could inform the decision-maker that we are normalizing local priorities, and elicit the appropriate criteria weights for normalized priorities. And if we added or deleted an alternative we could, recognizing that our previous criteria weights are now inappropriate, elicit new weights.

In this context, what AHP terms "criteria importances" are really merely scaling factors which reduce all local priorities to common units so that they can be added up. The necessary magnitudes of the scaling factors depends only on the relative magnitudes of the units of the composing scales, and not on some elusive notion of criteria importance.

The solution

Fortunately, there are many solutions to the problem of unwarranted rank reversals and priority shifts. These solutions share a common element - each requires an interpretation of the importance of criteria which is tied to a particular alternative or set of alternatives. Referenced AHP, which was first articulated in [8], although not called there by that name, is developed in [9]. So called "linking pin methods" are developed in [9, 10]. Finally, Saaty's supermatrix approach is detailed in [2]. I will not repeat the derivation of any of these approaches here, but will instead briefly outline how they would each play a game suggested by my colleague, Eng Choo.

A computer is programmed to value alternatives as follows. Each alternative is described by a 3-tuple of descriptions on each of three criteria, with each description corresponding to a ratio scaled value/utility. These values are on commensurate scales (i.e., inthe same units). In establishing the overall value of any particular alternative the computer adds the values corresponding to the elements in its 3-tuple. For example, the computer may store descriptions of houses, based on location, size and condition. Each location, size and condition has a corresponding value, with the overall value of each house determined by the sum of its component values.

The game is played as follows. Suppose there are 10 alternatives in the choice set. The computer picks some subset of alternatives and challenges you to find the relative values of this subset. You may question the computer, but all questions, as in AHP, must be on the basis of pairwise comparisons. The computer will respond truthfully. You are not permitted, however, to ask questions concerning overall value directly. When you have completed your questionning, the computer will generate another subset of alternatives for you to evaluate. The object of the game is for you to discover the relative values of the 10 alternatives. The structure of the game implies that any two alternatives must have a constant relative value across subsets, including the subset consisting of all 10 alternatives.

In the context of the house example, suppose Table 1 represents a partial list of values on the three criteria.

Table 1 House values on three criteria

House	Location	Value	Size	Value	Condition	Value	Overall Value
A	West side	20	1700 sq ft	5	Good	5	30
B	East side	4	2000 sq ft	10	Fair	1	15
C	West side	20	1200 sq ft	2	Excellent	8	30
etc.							

We might first examine this table according to the criterion proposed by Harker and Vargas [2] who state:

> An important assumption underlying the use of the Principle of Hierarchic Composition is that the weights of the criteria are independent of the alternatives considered. If this assumption

is violated, then the system with feedback approach must be used. The system with feedback approach is the supermatrix.

If I understand this criterion, the assumption is violated in this instance. For example, if we compare location and size from the perspective of house A, location is 4 times as important as size. But if we make the same comparison from the perspective of house B, size is 2.5 times as important as location. Thus standard AHP cannot be applied to applied to this problem.

But then when does standard AHP apply? In the above table, the only condition under which the criterion would not be violated is when the columns are proportional to one another (e.g., every house has location worth 5 times as much as size). But, if that were the case, we would need to estimate only one vector of local priorities to solve the problem, since local priorities and synthesized priorities would be identical. *Thus the Harker and Vargas criterion suggests that standard AHP applies only to problems which can be reduced to a single criterion.*

Now, let us play the game. Suppose the computer presents the first two houses on the list as the first subset for estimation, and then presents the first three houses on the list as the second subset for estimation. A successful procedure should yield relative priorities such that house A is valued as twice as much as house B on the first subset, and houses A and C are each valued twice as much as house B on the second subset. The supermatrix yields the correct answer. So do two techniques which have been termed "Referenced AHP" and "Linking Pin AHP" [10]. I demonstrate below how these latter two techniques can be employed to play the game.

Referenced AHP

As in conventional AHP we ask for paired comparisons on the alternatives under each criterion, and normalize the eigenvector of the resulting paired comparison matrix to sum to one. On the subset of size 2 this yields local priorities as follows:

Location	Size	Condition
$\begin{bmatrix} 5/6 \\ 1/6 \end{bmatrix}$	$\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$	$\begin{bmatrix} 5/6 \\ 1/6 \end{bmatrix}$

Referenced AHP requires that criteria importance be estimated by comparing the mean (or total) of the values under each criterion. Thus the computer would be asked questions like "Which has more value, the total location value of the two houses or the total size value of the two houses, and by how many times?" The computer would compare the total location value (24) and total size value (15) and respond "the total location value is 1.6 times the total size value". The questionner would complete the paired comparison matrix by comparing location with condition, and size with condition. The resulting vector of criteria importances would be:

$$\begin{bmatrix} 24/45 \\ 15/45 \\ 6/45 \end{bmatrix}$$

and synthesized priorities would be

$$y_A = 24/45x5/6 + 15/45x1/3 + 6/45x5/6 = 30/45$$
$$y_B = 24/45x1/6 + 30/45x2/3 + 39/45x1/6 = 15/30$$

which correspond to the ratio in Table 1.

Now suppose that alternatives A, B and C comprise the subset. Local priorities would be:

Location Size Condition

$$\begin{bmatrix} 5/11 \\ 1/11 \\ 5/11 \end{bmatrix} \begin{bmatrix} 5/17 \\ 10/17 \\ 2/17 \end{bmatrix} \begin{bmatrix} 5/14 \\ 1/14 \\ 8/14 \end{bmatrix}$$

In establishing criteria priorities, the total values under location, size and condition become 44, 17 and 14 respectively. Paired comparisons of these values would establish the vector of criteria priorities to be:

$$\begin{bmatrix} 44/75 \\ 17/75 \\ 14/75 \end{bmatrix}$$

and synthesized priorities would be:

$$y_A = 44/75x5/11 + 17/75x5/17 + 14/75x5/14 = 30/75$$
$$y_B = 44/75x1/11 + 17/75x10/17 + 14/75x1/14 = 15/75$$
$$y_C = 44/75x5/11 + 17/75x2/17 + 14/75x8/14 = 30/75$$

which correspond to the ratio in Table 1.

It is important to note that criteria priorities are entirely determined by the particular alternatives in the choice subset. As in the supermatrix, we have a system with feedback. Thus criteria priorities shift from the first subset evaluation to the second. In this sense, there is no answer to the abstract question, "which is more important in house choice, location or size, and by how much?"

Linking pins

A more straightforward approach to the problem is to employ the method of "linking pins". After local priorities are established, if we could estimate the ratio of the value of any one of the alternatives on a criterion against the value of that same alternative (or any other alternative on another criterion, we can reduce both scales to common units. Under this method, normalization of local priorities is to unity on the linking values. This is illustrated below for the subset consisting of A, B, and C.

We arbitrarily choose to link via alternative A. Thus the local priority vectors become:

Location	Size	Condition
$\begin{bmatrix} 1 \\ 1/5 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 2 \\ 2/5 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1/5 \\ 8/5 \end{bmatrix}$

The computer is now asked for paired comparisons of the value of the location of house A to the value of the size of house A (location 4x size); the value of the location of house A to the value of the condition of house A (location 4x condition); and the value of the size of house A to the value of the condition of house A (size 1x condition). It might be noted that these comprise a subset of the questions required for the supermatrix.

The vector of criteria priorities generated by these paired comparisons is:

$$\begin{bmatrix} 4/6 \\ 1/6 \\ 1/6 \end{bmatrix}$$

and synthesized priorities would be:

$$y_A = 4/6x1 + 1/6x1 + 1/6x1 = 1$$

$$y_B = 4/6x1/5 + 1/6x2 + 1/6x1/5 = 1/2$$

$$y_C = 4/6x1 + 1/6x2/5 + 1/6x8/5 = 1$$

which, again, corresponds to the ratios in Table 1. It should be evident that one need not provide other estimates of criterion priorities for any other subsets containing alternative A.

Summary and conclusions

I have argued that rank reversals which occur because of the addition of an alternative are symptomatic of an underlying problem in AHP. Because so much of [10] was devoted to the failure of conventional AHP to correctly estimate problems with known answers [10], I have not repeated these arguments here, but have instead concentrated on addressing the arguments of proponents of conventional AHP. I have also argued that we know the cause of rank reversals and relative priority shifts. Finally, we know how to modify AHP so that rank reversals and relative priority shifts do not occur.

It appears surprising to me that proponents of AHP would argue the legitimacy of rank reversals, and at the same time introduce axioms designed to eliminate rank reversals. Thus we have an axiom which excludes copies of existing alternatives, and an axiom which requires the choice set to be complete [2, 5]. But such axioms eliminate symptoms, not causes. A pity, since the AHP is so easily modified to correct the underlying problem.

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