MATHEMATICAL PROPERTIES OF DOMINANT AHP AND CONCURRENT CONVERGENCE METHOD

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Summary: *This study discusses the mathematical structure of the dominant AHP and the concurrent convergence method that which originally developed by Kinoshita and Nakanishi. They introduced a new concept of a regulating alternative into an analyzing tool for a simple evaluation problem with a criterion set and an alternative set. Although the original idea of the dominant AHP and the concurrent convergence method is unique, the dominant AHP and the concurrent convergence method are not sufficiently analyzed in mathematical theory. This study shows that the dominant AHP consists of a pair of evaluation rules satisfying a certain property of overall evaluation vectors. This study also shows that the convergence of concurrent convergence method is guaranteed theoretically.*

1. Introduction

AHP is a flexible decision making system that can deal with the subjective judgments of a decision maker. Numerously successful applications have been reported in this field (Saaty, 1980). In AHP, the decision maker identifies an ambiguous evaluation problem into a hierarchy structure within the evaluation goal, criteria and alternatives, each of which corresponds to a node of the hierarchy. The hierarchy with a top, middle and bottom structure usually consists of three levels, the goal, the criteria, and the alternatives, respectively. This study also discusses the three-level hierarchy. Directed arcs of the hierarchy form a parents-child relationship among the nodes and the existence of a pair of parents-child nodes means that the decision maker judges the relative importance of the child-nodes from the parents-node. That is, for example, directed arcs from the top level to the middle level indicate the decision maker's judgment on the relative importance of all criteria from the goal. Saaty (Saaty, 1980) proposes that in this three-level hierarchy the decision maker firstly judges the relative importance of the criteria from the goal and secondarily judges that of the alternative from the criteria. Judgments of the relative importance are expressed numerically, which are called evaluation values. Let I and J be a set of alternatives and that of criteria, respectively, and denote their cardinalities by I and J, respectively. Then we have a total of $|I|\times(J+1)$ evaluation values in the three-level hierarchy. By plotting a set of evaluation values on the arcs of hierarchy, the hierarchy becomes a tree of a network with the directed arcs. In the original AHP, the evaluation value of a child-node from a parents-node is quantified under the assumption that the decision maker compares all pairs between distinct two children of the parents.

Kinoshita and Nakanishi (Kinoshita and Nakanishi, 1999) focus on the following empirical result: When the decision maker evaluates relative importance of the criteria from the goal, he/she focuses on a specific alternative and refers to relative importance of the criteria from the specific alternative. The specific alternative is called the regulating alternative. Kinoshita and Nakanishi (Kinoshita and Nakanishi, 1999) assume that if there exists exactly one regulating alternative then the relative importance of the criteria from the regulating alternative determines that from other alternatives. If there

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exists only one regulating alternative in the alternative set, then the regulating alternative is called the dominant one and they implement the assumption into the dominant AHP. The mathematical description of the dominant AHP is as follows:

Step 0: The decision maker selects a regulating alternative from the alternative set I. Let alternative k be the regulating alternative.

Step 1: From the viewpoint of every criterion $j \in J$, the decision maker evaluates the relative importance of all alternatives and quantifies the evaluation values of all alternatives. Let a_{ij} be the evaluation value of the alternative i from the criterion j and let A be an $|J| \times |J|$ evaluation matrix whose (i, j) element is a_{ij}.

Step 2: From the viewpoint of the regulating alternative k, the decision maker evaluates the relative importance of all criteria and quantifies the evaluation values of all criteria by such as the eigenvalue method for a pairwise comparison matrix of the criteria. Let $\boldsymbol{b}^k$ be a $|J|$-dimensional vector whose jth element is the evaluation value of the criterion j from the regulating alternative k.

Step 3: Let A_k be a $|J| \times |J|$ diagonal matrix whose (i, j) element is a_{kj}. Calculate $AA_k^{-1}\boldsymbol{b}^k$ and define the ith element of $AA_k^{-1}\boldsymbol{b}^k$ as the overall evaluation value of alternative i.

Suppose that the alternative k is the dominant one. Let $\hat{\boldsymbol{b}}^i$ be a $|J|$-dimensional vector whose jth element is the unknown evaluation value of the criterion j from the alternative $i \neq k$, and let A_i be a diagonal matrix whose (j, j) element is a_{ij}. Then, Kinoshita and Nakanishi (Kinoshita and Nakanishi, 1999) propose a following evaluation rule under their assumption:

$$\hat{\boldsymbol{b}}^i = \frac{A_i A_k^{-1} \boldsymbol{b}^k}{\boldsymbol{e}^{\mathrm{T}} A_i A_k^{-1} \boldsymbol{b}^k} \tag{1}$$

for all $i \in I \setminus \{k\}$, where $\boldsymbol{e}$ is all one vector and $^{\mathrm{T}}$ stands for the transpose operation. They define

$$AA_i^{-1}\hat{\boldsymbol{b}}^i \tag{2}$$

as the overall evaluation vector derived from the alternative i and they point out that $AA_i^{-1}\hat{\boldsymbol{b}}^i$ coincides (except for a scalar multiple) with $AA_k^{-1}\boldsymbol{b}^k$ for all $i \in I \setminus \{k\}$. Therefore, they assert that the overall evaluation vector $AA_k^{-1}\boldsymbol{b}^k$ is valid.

In order to deal with an additional data to A, they relax their assumption and extend single regulating alternative to multiple ones. Let K be an index set of regulating alternatives, then $\boldsymbol{b}^k$ of the regulating alternative $k \in K$, can be given by Step 2 and $|K|$ types of A, say $\{A^{(k)} \mid k \in K\}$, can be given by Step 1. They assume that the relative importance of criteria from every alternative is an aggregately relative importance of criteria from all regulating alternatives. Under the assumption they develop a two-stage procedure as follows: First, merge $\{A^{(k)} \mid k \in K\}$ into a positive matrix A by an appropriate method (Kinoshita and Nakanishi, 1999). Next, apply the evaluation rule (1) to estimating $\hat{\boldsymbol{b}}^i$ from relative importance $\boldsymbol{b}^k$ of each regulating alternative $k \in K$. Hence, calculate

$$A_i A_k^{-1}\boldsymbol{b}^k \tag{3}$$

for all $i \in I$ and all $k \in K$ and generate $\hat{\boldsymbol{b}}^i$ from $\{A_i A_k^{-1}\boldsymbol{b}^k \mid k \in K\}$ by an iterative procedure (Kinoshita and Nakanishi, 1999), for every $i \in I$.

The two-stage procedure is called the concurrent convergence method in (Kinoshita and Nakanishi, 1999). However, convergence of the iterative procedure in the second stage has not been guaranteed theoretically and it is still an open problem (Takahashi, 1998). Kinoshita and Nakanishi (Kinoshita and Nakanishi, 1999) observe in a numerical example that the concurrent convergence method provides a limit point set $\left\{\hat{\boldsymbol{b}}^i \middle| i \in I\right\}$ and that $AA_1^{-1}\hat{\boldsymbol{b}}^1$ coincides (except for a scalar multiple) with $AA_i^{-1}\hat{\boldsymbol{b}}^i$ for all $i \in I$. The latter observation arises in both the dominant AHP and the concurrent convergence method. It is called the consistency property.

The first aim of this study is to show that some pairs of evaluation rules satisfy the consistency property other than the pair of (1) and (2) in the dominant AHP. The second aim is to prove the convergence of the iterative procedure in the concurrent convergence method. This study contributes the mathematical foundations and generalizations of the dominant AHP and the concurrent convergence method.

This paper has five sections Section 2 discusses the mathematical structure of the dominant AHP, especially the pair of evaluation rules (1) and (2). Section 3 shows the convergence and the consistency property of the concurrent convergence method. The final section is a brief conclusion. We outline some future extensions as well.

2. Mathematical structure of the dominant AHP Model

In this section, we discuss mathematical properties of the overall evaluation vector $AA_k^{-1}\boldsymbol{b}^k$ and alternative *i*'s evaluation vector $\hat{\boldsymbol{b}}^i$ of the criteria that is estimated by regulating alternative k. (Note that A_i is well defined for all $i \in I$ since A is a positive matrix.) We only focus on the directions of the overall evaluation vectors, $AA_k^{-1}\boldsymbol{b}^k$ and $AA_i^{-1}\hat{\boldsymbol{b}}^i$, and the evaluation vectors of criteria, $\boldsymbol{b}^k$ and $\hat{\boldsymbol{b}}^i$. The overall evaluation vector is to indicate the relative importance of alternatives and its length has no information concerning alternatives. So, if a vector a coincides except for a scalar multiple with a vector b, we say that a has the same direction as b. In order to express the mathematical properties of the dominant AHP, we introduce two linear transformations, $\mathrm{B}_i^k(\bullet)$ and $V_i(\bullet)$ as follows: For a $|J|$-dimensional vector $\boldsymbol{b}$, we define $V_i(\boldsymbol{b}) = AA_i^{-1}\boldsymbol{b}$ and $\mathrm{B}_i^k(\boldsymbol{b}) = A_iA_k^{-1}\boldsymbol{b}$ for all $i,k \in I$. Then, the overall evaluation vector by the evaluation rule (2) is given by the function value $V_i(\hat{\boldsymbol{b}}^i)$.

Lemma 1 *Suppose that* $\hat{\boldsymbol{b}}^i$ *is defined by (1) for all* $i \in I \setminus \{k\}$*, then*

$$\hat{\boldsymbol{b}}^i = \frac{B_i^k(\boldsymbol{b}^k)}{\boldsymbol{e}^{\mathrm{T}} A_i A_k^{-1}\boldsymbol{b}^k} \tag{4}$$

That is, $\hat{b}^i$ *has the same director as* $B_i^k(\boldsymbol{b}^k)$ *for all* $i \in I \setminus \{k\}$.

Proof: $$\hat{\boldsymbol{b}}^i = \frac{A_iA_k^{-1}\boldsymbol{b}^k}{\boldsymbol{e}^{\mathrm{T}} A_i A_k^{-1}\boldsymbol{b}^k} = \frac{B_i^k(\boldsymbol{b}^k)}{\boldsymbol{e}^{\mathrm{T}} A_i A_k^{-1}\boldsymbol{b}^k}$$

for all $i \in I \setminus \{k\}$. ■

Then we summarize the consistency property of the dominant AHP as follows:

Theorem 2 Let $\boldsymbol{b}$ be a $|J|$-dimensional vector, then $V_k(\boldsymbol{b}) = V_i(B_i^k(\boldsymbol{b}))$ for all $i,k \in I$. Suppose that $\hat{\boldsymbol{b}}^i$ is defined by (1) for all $i \in I \setminus \{k\}$, then $V_k(\boldsymbol{b}^k)$ has the same direction as $V_i(\hat{\boldsymbol{b}}^i)$.

Proof: For every $|J|$-dimensional vector $\boldsymbol{b}$,

$$V_i(B_i^k(\boldsymbol{b})) = AA_i^{-1}(A_i A_k^{-1}\boldsymbol{b}) = AA_k^{-1}\boldsymbol{b} = V_k(\boldsymbol{b}) \tag{5}$$

for all $i, k \in I$. It follows from Lemma 1 and (5) that

$$V_i\left(\hat{\boldsymbol{b}}^i\right) = V_{\mathrm{i}}\left(\frac{B_i{}^k(\boldsymbol{b}^k)}{\boldsymbol{e}^{\mathrm{T}} A_i A_k^{-1} \boldsymbol{b}^k}\right) = \frac{A A_i^{-1} B_i^k(\boldsymbol{b}^k)}{\boldsymbol{e}^{\mathrm{T}} A_i A_k^{-1} \boldsymbol{b}^k} = \frac{V_i(B_i^k(\boldsymbol{b}^k))}{\boldsymbol{e}^{\mathrm{T}} A_i A_k^{-1} \boldsymbol{b}^k} = \frac{V_k(\boldsymbol{b}^k)}{\boldsymbol{e}^{\mathrm{T}} A_i A_k^{-1} \boldsymbol{b}^k}$$

for all $i \in I \setminus \{k\}$. ■

From Theorem 2, Kinoshita and Nakanishi (Kinoshita and Nakanishi, 1999) mention that the pair of evaluation rules (1) and (2) provides a consistent overall evaluation vector among all alternatives.

Under the assumption that alternative *i* has the evaluation vector $\hat{\boldsymbol{b}}^i$ of criteria, we apply (1) to estimating alternative *k*'s evaluation vector of criteria from $\hat{\boldsymbol{b}}^i$ and then obtain $\boldsymbol{b}^k$. Hence, $B_i^k(\bullet)$ can be considered as an inverse function of $B_k^i(\bullet)$ in the sense as follows:

Theorem 3 *Let $\boldsymbol{b}$ be a $|J|$-dimensional vector, then $B_i^k\left(B_k^i(\boldsymbol{b})\right) = \boldsymbol{b}$ for all $i, k \in I$. Suppose that $\hat{\boldsymbol{b}}^i$ is defined by (1) for all $i \in I \setminus \{k\}$, then $B_k^i(\hat{\boldsymbol{b}}^i)$ has the same direction as $\boldsymbol{b}^k$.*

Proof: It follows from definitions of $B_k^i(\bullet)$ and $B_i^k(\bullet)$ that

$$B_i^k\left(B_k^i(\boldsymbol{b})\right) = B_i^k\left(A_k A_k^{-1} \boldsymbol{b}\right) = A_i A_k^{-1} A_k A_i^{-1} \boldsymbol{b} = \boldsymbol{b}\,. \tag{6}$$

Since

$$\hat{\boldsymbol{b}}^i = \frac{A_i A_k^{-1} \boldsymbol{b}^k}{\boldsymbol{e}^{\mathrm{T}} A_i A_k^{-1} \boldsymbol{b}^k} = B_i^k\left(\frac{\boldsymbol{b}^k}{\boldsymbol{e}^{\mathrm{T}} A_i A_k^{-1} \boldsymbol{b}^k}\right),$$

it follows from (5) that

$$B_i^k\left(\hat{\boldsymbol{b}}^i\right) = \frac{\boldsymbol{b}^k}{\boldsymbol{e}^{\mathrm{T}} A_i A_k^{-1} \boldsymbol{b}^k}$$

for all $i \in I \setminus \{k\}$. ■

3. Mathematical structure of the concurrent convergence method

In this section, we consider the case that there exist several regulating alternatives, that is the case of $|\mathrm{K}| \geq 2$. The concurrent convergence method begins with merging $\left\{A^{(k)} \middle| k \in K\right\}$ to generate a common evaluation matrix A for all alternatives. This is the first stage of the concurrent convergence method. Then, we go to the following initial step of the second stage as follows :

Algorithm 0

Step 0 : For a given set of the evaluation vectors of criteria, $\left\{\boldsymbol{b}^k \middle| k \in K\right\}$, in the first stage, let

$$\boldsymbol{b}_0^k := \boldsymbol{b}^k \tag{7}$$

for all $k \in K$. Let $t := 0$ and go to **Step 1**.

Step 1: Let

$$\boldsymbol{b}_{t+1}^i := \frac{1}{|K|} \sum_{k \in K} \frac{A_i A_k^{-1} \boldsymbol{b}_t^k}{\boldsymbol{e}^{\mathrm{T}} A_i A_k^{-1} \boldsymbol{b}_t^k} \tag{8}$$

for all $i \in I$.

Step 2: If $\max\left\{\left\|\boldsymbol{b}_{t+1}^{i} - \boldsymbol{b}_{t}^{i}\right\| \mid i \in I\right\} = 0$ then set $\bar{\boldsymbol{b}}^{i} = \boldsymbol{b}_{t+1}^{i}$ and stop. Otherwise, update $t := t+1$ and go to **Step 1.**

Kinoshita and Nakanishi (Kinoshita and Nakanishi, 1999) report in some numerical experiments that Algorithm 0 has a limit point set $\{\bar{\boldsymbol{b}}^{i} \mid i \in I\}$ such that $AA_i^{-1}\bar{\boldsymbol{b}}^{i}$ has the same direction as $AA_l^{-1}\bar{\boldsymbol{b}}^{i}$ for all $i, l \in I$.

To prove the observation above, we consider the following Algorithms 1 which is simplified from Algorithm 0.

Algorithm 1

Step 0: For all $k \in K$, let

$$\boldsymbol{p}_0^k := A_k^{-1}\boldsymbol{b}^k. \tag{9}$$

Set $t := 0$.

Step 1: For all $k \in K$, let

$$\boldsymbol{p}_{t+1}^{k} := \frac{1}{|K|}\sum_{l \in K}\frac{\boldsymbol{p}_t^l}{\boldsymbol{e}^{\mathrm{T}} A_k \boldsymbol{p}_t^l}. \tag{10}$$

Step 2 : If $\max_{k\in K}\left\|\boldsymbol{p}_{t+1}^{k} - \boldsymbol{p}_{t}^{k}\right\| = 0$, them for all $k \in K$, let $\bar{\boldsymbol{p}}^{k} := \boldsymbol{p}_{t+1}^{k}$ and stop.

Otherwise, set $t := t+1$ and go to **Step 1 .**

We can correspond $\{\boldsymbol{p}_t^k \mid t = 0,1,2,.....\}$ of Algorithm 1 to $\{\boldsymbol{b}_t^k \mid t = 0,1,2,.....\}$ of Algorithm 0 as follows:

Lemma 8 *The equation*

$$A_k^{-1}\boldsymbol{b}_t^k = \boldsymbol{p}_t^k \tag{11}$$

holds for all $k \in K$ *and* $t = 0,1,...$, *and*

holds for all $i \in I$ *and* $t = 1,2,...$. *If Algorithm 0 is convergent finitely , then so Algorithm 1, and vice versa.*

Proof: We will show (11) by induction of iteration *t*. At iteration $t = 0$, $A_k^{-1}\boldsymbol{b}_0^k = \boldsymbol{p}_0^k$ follows directly from (7) and (9) for all $k \in K$. We assume that at iteration s $A_k^{-1}\boldsymbol{b}_s^k = \boldsymbol{p}_s^k$ holds for all $k \in K$. Then it follows (8) and (10) that

$$A_k^{-1}\boldsymbol{b}_{s+1}^k = A_k^{-1}\frac{1}{|K|}\sum_{l\in K}\frac{A_k A_l^{-1}\boldsymbol{b}_s^l}{\boldsymbol{e}^{\mathrm{T}} A_k A_l^{-1}\boldsymbol{b}_s^l} = \frac{1}{|K|}\sum_{l\in K}\frac{\boldsymbol{p}_s^l}{\boldsymbol{e}^{\mathrm{T}} A_k \boldsymbol{p}_s^l} = \boldsymbol{p}_{s+1}^k$$

for all $k \in K$. Therefore, we complete (11) for all $k \in K$ and $t = 0,1,...$. This means from (8) that

$$\boldsymbol{b}_t^i = \frac{1}{|K|}\sum_{i\in K}\frac{A_i \boldsymbol{p}_t^l}{\boldsymbol{e}^{\mathrm{T}} A_i \boldsymbol{p}_t^l}$$

for all $i \in I$ and $t = 1,2,...$.

When Algorithm 0 stops at iteration t, we have $b_{t+1}^i = b_t^i$ for all $i \in I$ and hence, $A_k^{-1}\boldsymbol{b}_{t+1}^k = A_k^{-1}\boldsymbol{b}_t^k$ for all $k \in K$. Therefore, Algorithm 1 stops at iteration t. On the contrary, if Algorithm 1 stops at iteration *t*, we have $A_k^{-1}\boldsymbol{b}_{t+1}^k = A_k^{-1}\boldsymbol{b}_t^k$. This implies that $\boldsymbol{b}_{t+1}^k = \boldsymbol{b}_t^k$. ■

From Lemma 8 we only discuss the convergence of Algorithm 1 instead of Algorithm 0.

Lemma 9 *The vector* $\boldsymbol{p}_t^k$ *is a positive vector for every* $k \in K$ *and* $t = 0,1,...$.

Proof: Note that $\boldsymbol{b}_0^k$ is a positive vector for all $k \in K$. Since every diagonal element of the diagonal matrix A_k is positive, it follows from Lemma 8 that $\boldsymbol{p}_0^k$ is also a positive vector for all $k \in K$. Assume that $\boldsymbol{p}_t^k$ is a positive vector for all $k \in K$, then it follows from (10) that $\boldsymbol{p}_{t+1}^k$ is positive for all $k \in K$.

■

Lemma 10 *For every* $k \in K$ *and* $t = 0,1,\ldots$, *the following equation holds*

$$\boldsymbol{e}^{\mathrm{T}} A_k \boldsymbol{p}_t^k = 1. \tag{12}$$

Proof: It follows from (8) that

$$\boldsymbol{e}^{\mathrm{T}} \boldsymbol{b}_{t+1}^k = \frac{1}{|K|} \sum_{l \in K} \frac{\boldsymbol{e}^{\mathrm{T}} A_k A_l^{-1} \boldsymbol{b}_t^l}{\boldsymbol{e}^{\mathrm{T}} A_k A_k^{-1} \boldsymbol{b}_t^l} = 1 \tag{13}$$

for all $k \in K$ *and* $t = 0,1,\ldots$. Since $\boldsymbol{e}^{\mathrm{T}} \boldsymbol{b}_0^k = 1$. For all $k \in K$, it follows from (13) that

$$\boldsymbol{e}^{\mathrm{T}} A_k \boldsymbol{p}_t^k = \boldsymbol{e}^{\mathrm{T}} A_k A_k^{-1} \boldsymbol{b}_t^k = \boldsymbol{e}^{\mathrm{T}} \boldsymbol{b}_t^k = 1.$$

for all $k \in K$ *and* $t = 0,1,\ldots$. ■

Lemmas 9 and 10 imply that $\left\{\boldsymbol{p}_t^k \,\middle|\, t = 0,1\ldots\right\}$ is a bounded set in the positive orthant for all $k \in K$.

Lemma 11 *For all* $k \in K$ *and* $t = 0,1,\ldots$,

$$\frac{\boldsymbol{p}_{t+1}^k}{\boldsymbol{e}^{\mathrm{T}} A_k \boldsymbol{p}_{t+1}^k} = \boldsymbol{p}_k^{t+1}.$$

Proof: The assertion is directly from Lemma 10. ■

Consider the following convex cone

$$\mathrm{Cone}(\{\boldsymbol{p}_{t+1}^k \mid k \in K\}) = \{\boldsymbol{p} \mid \boldsymbol{p} = \sum_{k \in K} \mu_k \boldsymbol{p}_{t+1}^k \text{ and } \mu_k \geq 0 \text{ for all } k \in K\}, \tag{14}$$

which is generated by the vectors $\{\boldsymbol{p}_{t+1}^k \mid k \in K\}$. For a set *D* we denote the relative interior and relative boundary of *D* by ri*D* and bd*D*, respectively.

Lemma 12 *Let* $\boldsymbol{R}$ *be an extreme ray set of* $\mathrm{Cone}(\{\boldsymbol{p}_t^k \mid k \in K\})$. *If* dim$\boldsymbol{R}$ *= 1, then Algorithm 1 stops. Otherwise, for every* $k \in K$ *and* $t = 0,1,\ldots$

$$\boldsymbol{p}_{t+1}^k \in \mathrm{riCone}(\{\boldsymbol{p}_{\mathrm{t}}^k \mid k \in K\}) \tag{15}$$

$$\boldsymbol{p}_{t+1}^k \notin \boldsymbol{R} \tag{16}$$

Proof: Note that Cone($\boldsymbol{R}$) = $\mathrm{Cone}(\{\boldsymbol{p}_t^k \mid k \in K\})$.

Firstly we consider the case of dim$\boldsymbol{R}$ =1. Since $p_k^t \in \mathrm{Cone}(\boldsymbol{R})$ for all $k \in K$, there exist positive numbers μ_{lk} for all $i, k \in K$ such that $\boldsymbol{p}_t^l = \mu_{lk} \boldsymbol{p}_t^k$. This means from (10) and Lemma 11 that

$$\boldsymbol{p}_{t+1}^k = \frac{1}{|K|} \sum_{l \in K} \frac{\boldsymbol{p}_t^l}{\boldsymbol{e}^{\mathrm{T}} A_k \boldsymbol{p}_t^l} = \frac{1}{|K|} \sum_{l \in K} \frac{\mu_{lk} \boldsymbol{p}_t^k}{\mu_{lk} \boldsymbol{e}^{\mathrm{T}} A_k \boldsymbol{p}_t^k} = \frac{1}{|K|} \sum_{l \in K} \frac{\boldsymbol{p}_t^k}{\boldsymbol{e}^{\mathrm{T}} A_k \boldsymbol{p}_t^k} = \frac{\boldsymbol{p}_t^k}{\boldsymbol{e}^{\mathrm{T}} A_k \boldsymbol{p}_t^k} = \boldsymbol{p}_t^k$$

for all $k \in K$. This satisfies the stopping criterion of Step 3.

Next, we consider $\dim \boldsymbol{R} \geq 2$. Then, it follows that $\mathrm{riCone}(\boldsymbol{R}) \subset \mathrm{Cone}(\boldsymbol{R})$ and that $\mathrm{riCone}(\boldsymbol{R}) \cap \boldsymbol{R} = \varnothing$. Note that if $\nu_k > 0$ then $(\sum_{k \in K} \nu_k \boldsymbol{p}_t^k) \in \mathrm{riCone}(\boldsymbol{R})$. It follows from Lemma 9 that $\boldsymbol{e}^{\mathrm{T}} A_k \boldsymbol{p}_t^l > 0$ for all $k, l \in K$. This means from (10) that

$$\boldsymbol{p}_{t+1}^k \in \mathrm{riCone}(\boldsymbol{R})$$

for all $k \in K$. This means that $\boldsymbol{p}_{t+1}^k \notin \boldsymbol{R}$ for all $k \in K$. ■

Lemma 13

$$\text{Cone}(\{\boldsymbol{p}_{t+1}^k \mid k \in K\}) \subset \text{Cone}(\{\boldsymbol{p}_t^k \mid k \in K\}) \quad \text{and}$$

$$\text{Cone}(\{\boldsymbol{p}_{t+1}^k \mid k \in K\}) \cap \text{bdCone}(\{\boldsymbol{p}_t^k \mid k \in K\}) = \{\textit{the origin}\}$$

for $t = 0,1,\ldots$.

Proof: It is directly from Lemma 12. ■

Lemma 13 means that Cone $(\{\boldsymbol{p}_t^k \mid k \in K\})$ shrinks monotonically for t = 0,1…. We assume that Algorithm 1 generates an infinite sequence of points $\{\boldsymbol{p}_t^k \mid k \in K\}$. Let $S^k = \{A_k^{-1}\boldsymbol{b} \mid \boldsymbol{b} \geq 0 \text{ and } \boldsymbol{e}^{\mathrm{T}}\boldsymbol{b} = 1\}$ for all $k \in K$,then S^k is compact and the product set $\prod S^k$ is also compact. This implies the following lemma.

Lemma 14 *For all* $k \in K$ *and* $t = 0,1,\ldots$,

$$\boldsymbol{p}_t^k \in S^k .$$

Moreover, there exist an index set $T \subseteq \{1,2,\ldots\}$ *and an accumulation point* $\hat{\boldsymbol{p}}^k$ *for all* $k \in K$ *such that*

$$\lim_{t \in T, t \to \infty} \boldsymbol{p}_t^k = \hat{\boldsymbol{p}}^k .$$

Proof: It follows from Lemma 10 that $\boldsymbol{p}_t^k \in S^k$ for all $k \in K$ and t=0,1…. Since the set S^k is compact for all $k \in K$, the product set $\prod S^k$ is also compact. Therefore, $\lim_{t \in T, t \to \infty} \boldsymbol{p}_t^k = \hat{\boldsymbol{p}}^k$ for all $k \in K$. ■

Lemma 15 *Suppose that the index set* $T \subseteq \{1,2,\ldots\}$ *and* $|K|$ *points* $\{\hat{\boldsymbol{p}}^k \mid k \in K\}$ *satisfy*

$$\lim_{t \in T, t \to \infty} \boldsymbol{p}_t^k = \hat{\boldsymbol{p}}^k . \tag{17}$$

for all $k \in K$ *, then* Cone $(\{\hat{\boldsymbol{p}}^k \mid k \in K\})$ *is a half-line.*

Proof: Note that Cone$(\{\hat{\boldsymbol{p}}^k \mid k \in K\}) \subset$ Cone $(\{\hat{\boldsymbol{p}}_t^k \mid k \in K\})$ for every $t \in T$, That is,

$$\text{Cone}(\{\hat{\boldsymbol{p}}^k \mid k \in K\}) \setminus \text{Cone}(\{\hat{\boldsymbol{p}}_t^k \mid k \in K\}) = \varnothing . \tag{18}$$

It follows from (10) that for every $t \in T$

$$\lim_{t \in T, t \to \infty} \boldsymbol{p}_{t+1}^k = \lim_{t \in T, t \to \infty} \sum_{l \in K} \frac{\boldsymbol{p}_t^l}{\boldsymbol{e}^{\mathrm{T}} A_k \boldsymbol{p}_t^l} = \sum_{l \in K} \frac{\hat{\boldsymbol{p}}^l}{\boldsymbol{e}^{\mathrm{T}} A_k \hat{\boldsymbol{p}}^l} . \tag{19}$$

It follows from Lemma 13 that $\hat{\boldsymbol{p}}^k \in \text{Cone}(\{\boldsymbol{p}_{t+1}^k \mid k \in K\})$ for every $t \in T$. Since $\text{Cone}(\{\boldsymbol{p}_{t+1}^k \mid k \in K\})$ is included in the positive orthant, we have $\boldsymbol{e}^{\mathrm{T}} A_k \hat{\boldsymbol{p}}^l > 0$ for all $k, l \in K$. Therefore, let $\tilde{\boldsymbol{p}}^l = \lim_{t \in T, t \to \infty} \boldsymbol{p}_{t+1}^l$, it follows from (19) that

$$\tilde{\boldsymbol{p}}^l \in \text{riCone}(\{\hat{\boldsymbol{p}}^k \mid k \in K\}) \tag{20}$$

for all $l \in K$. This means that $\text{Cone}(\{\tilde{\boldsymbol{p}}^k \mid k \in K\}) \subseteq \text{Cone}(\{\hat{\boldsymbol{p}}^k \mid k \in K\})$.

We will show $\text{Cone}(\{\tilde{\boldsymbol{p}}^k \mid k \in K\}) = \text{Cone}(\{\hat{\boldsymbol{p}}^k \mid k \in K\})$. Suppose that $\text{Cone}(\{\tilde{\boldsymbol{p}}^k \mid k \in K\}) \subset \text{Cone}(\{\hat{\boldsymbol{p}}^k \mid k \in K\})$ and let

$$B_\varepsilon(\boldsymbol{x}) = \{\boldsymbol{y} \mid \| \boldsymbol{x} - \boldsymbol{y} \| \leq \varepsilon\} \text{ and } G(\varepsilon) = \text{Cone}(\cup_{k \in K} B_\varepsilon(\tilde{\boldsymbol{p}}^k)) .$$

Since $\text{Cone}(\{\tilde{\boldsymbol{p}}^k \mid k \in K\})$ is a closed set, there exists a positive scalar $\bar{\varepsilon}$ such that

$$\text{Cone}(\{\hat{\boldsymbol{p}}^k \mid k \in K\}) \setminus G(\bar{\varepsilon}) \neq \varnothing . \tag{21}$$

It follows from (20) that there exists an index $t_1 \in T$ such that $\mathrm{Cone}\left(\left\{\boldsymbol{p}_{t_1+1}^k \middle| k \in K\right\}\right) \subseteq G(\bar{\varepsilon})$. Therefore, suppose that $t_2 \geq t_1 + 1$ and that $t_2 \in T$, then it follows from Lemma 13 that $\mathrm{Cone}\left(\left\{\boldsymbol{p}_{t_2}^k \middle| k \in K\right\}\right) \subset \mathrm{Cone}\left(\left\{\boldsymbol{p}_{t_1+1}^k \middle| k \in K\right\}\right) \subseteq G(\bar{\varepsilon})$. It follows from (21) that $\mathrm{Cone}\left(\left\{\hat{\boldsymbol{p}}^k \middle| k \in K\right\}\right) \setminus \mathrm{Cone}\left(\left\{\boldsymbol{p}_{t_2}^k \middle| k \in K\right\}\right) \neq \varnothing$. This is a contradiction for (18). Therefore, $\mathrm{Cone}\left(\left\{\tilde{\boldsymbol{p}}^k \middle| k \in K\right\}\right) = \mathrm{Cone}\left(\left\{\hat{\boldsymbol{p}}^k \middle| k \in K\right\}\right)$ and hence, it follows from (20) that $\mathrm{Cone}\left(\left\{\hat{\boldsymbol{p}}^k \middle| k \in K\right\}\right)$ is a half-line.

■

The following lemma guarantees the existence of a limit point of the infinite sequence $\{\boldsymbol{p}_t^k \mid t = 0,1,\ldots\}$ for all $k \in K$.

Theorem 16 *If Algorithm 1 repeats infinitely, there exist a half-line* $\boldsymbol{H}$ *and a limit point* $\hat{\boldsymbol{p}}^k$ *of* $\{\boldsymbol{p}_t^k \mid t = 0,1,\ldots\}$ *for all* $k \in K$ *such that*

$$\hat{\boldsymbol{p}}^k \in \boldsymbol{H} . \tag{22}$$

Hence, $\hat{\boldsymbol{p}}^k$ has the same direction as $\hat{\boldsymbol{p}}^l$ for all $k, l \in K$.

Proof: It is trivial from Lemmas 10, 13 and 15. ■

From the viewpoint of set convergence (Rockafellar and Wets, 1998), Lemma 16 implies that $\mathrm{Cone}\left(\left\{\boldsymbol{p}_t^k \middle| k \in K\right\}\right)$ converges on a half line of the positive orthant. When Algorithm 1 converges within finitely many iterations, the point set $\{\hat{\boldsymbol{p}}^k \mid k \in K\}$ of Stet 2 has the same property as stated in Lemma 16.

Lemma 17 Suppose that Algorithm 1 stops within finitely many iterations and let $\hat{\boldsymbol{p}}^k$ be defined by Step 2 of Algorithm 1 for all $k \in K$, then $\hat{\boldsymbol{p}}^k$ has the same direction as $\hat{\boldsymbol{p}}^l$ for all $k, l \in K$.

Proof: Suppose that Algorithm 1 stops at iteration t, then we have

$$\boldsymbol{p}_{t+1}^k = \boldsymbol{p}_t^k = \hat{\boldsymbol{p}}^k \tag{23}$$

for all $k \in K$. It follows from (10), Lemma 11 and (23) that

$$\begin{aligned}\hat{\boldsymbol{p}}^k = \boldsymbol{p}_{t+1}^k &= \frac{1}{|K|} \sum_{l \in K} \frac{\boldsymbol{p}_t^l}{\boldsymbol{e}^{\mathrm{T}} A_k \boldsymbol{p}_t^l} = \frac{1}{|K|} \sum_{l \neq k, l \in K} \frac{\boldsymbol{p}_t^l}{\boldsymbol{e}^{\mathrm{T}} A_k \boldsymbol{p}_t^l} + \frac{1}{|K|} \frac{\boldsymbol{p}_t^k}{\boldsymbol{e}^{\mathrm{T}} A_k \boldsymbol{p}_t^k} \\ &= \frac{1}{|K|} \sum_{l \neq k, l \in K} \frac{\boldsymbol{p}_t^l}{\boldsymbol{e}^{\mathrm{T}} A_k \boldsymbol{p}_t^l} + \frac{1}{|K|} \boldsymbol{p}_t^k = \frac{1}{|K|} \sum_{l \neq k, l \in K} \frac{\hat{\boldsymbol{p}}^l}{\boldsymbol{e}^{\mathrm{T}} A_k \hat{\boldsymbol{p}}^l} + \frac{1}{|K|} \hat{\boldsymbol{p}}^k\end{aligned}$$

for all $k \in K$. Therefore, we have

$$\left(|K| - 1\right) \hat{\boldsymbol{p}}^k = \sum_{l \neq k, l \in K} \frac{\hat{\boldsymbol{p}}^l}{\boldsymbol{e}^{\mathrm{T}} A_k \hat{\boldsymbol{p}}^l}$$

which means that $\hat{\boldsymbol{p}}^k \in \mathrm{Cone}\left(\left\{\hat{\boldsymbol{p}}^l \middle| l \in K, l \neq k\right\}\right)$. This implies that $\dim \mathrm{Cone}\left(\left\{\boldsymbol{p}^k \middle| k \in K\right\}\right) = 1$. Hence, $\hat{\boldsymbol{p}}^k$ has the same direction as $\hat{\boldsymbol{p}}^l$ for all $k, l \in K$. ■

By the above lemmas we can summarize the mathematical properties of the concurrent convergence method as follows:

Theorem 18 *The concurrent convergence method has a limit point set* $\left\{ \bar{\boldsymbol{b}}^i \mid i \in I \right\}$. *Let* $AA_i^{-1}\bar{\boldsymbol{b}}^i$ *be the overall evaluation vector of alternative* i, *then the overall evaluation vector of alternative i has the same direction as that of alternative l for all* $i, l \in I$.

Proof: The assertion follows directly from Lemma 8, 16 and 17. ■

4. Conclusion

This paper develops the mathematical foundations of the dominant AHP and a mechanism for the convergence of the concurrent convergence method. Hence, we show the mathematical description that the dominant AHP consists of a pair of simple evaluation rules (1) and (2) and that the pair of the rules provides the consistency property between regulating alternative's overall evaluation vector and other alternative's ones. Furthermore we discuss an extension of the evaluation rules (1) and (2) without violating the property. As stated in Example 1, one can apply the proposed evaluation rules to sensitive for the overall evaluation vector.

This paper shows the convergence of the concurrent convergence method whose $\boldsymbol{p}_{t+1}^i$ is fixed as the non-weighted average of $\left\{ \boldsymbol{p}_t^l / \left(\boldsymbol{e}^{\mathrm{T}} A_i \boldsymbol{p}_t^l \right) \mid l \in K \right\}$ in Step 1. By the same way as the proofs in section 3, we can guarantee the convergence of a variant concurrent convergence method whose $\boldsymbol{p}_{t+1}^i$ is given by a weighted average of $\left\{ \boldsymbol{p}_t^l / \left(\boldsymbol{e}^{\mathrm{T}} A_i \boldsymbol{p}_t^l \right) \mid l \in K \right\}$. Exploiting the convergence, one can extend the dominant AHP into an analyzing tool for an evaluation problem with a complex network structure (Kinoshita and Nakanishi, Submitted, Sekitani and Takahashi, 2001), interval AHP (Arbel and Vargas, 1992) and group AHP (Nakanishi and Kinoshita, 1998, Yamada and Sugiyama, 1997).

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