How to Make a Decision: The Analytic Hierarchy Process

Thomas L. Saaty
University of Pittsburgh

ABSTRACT

This paper, delivered as a keynote address at the conference, summarizes the basic concepts of the Analytic Hierarchy Process.

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1. Introduction

"You can't compare apples and oranges," the saying goes. But is this true? Consider a hungry person who likes both apples and oranges and is offered a choice between a large, red, pungent, juicy looking Washington State apple and an even larger, old and shrivelled, pale color orange with a soft spot. Which one is that person more likely to choose? Let us reverse the situation and offer the same person on the next day a small, deformed, unripe apple with a couple of worm holes and a fresh colored navel orange from California. Which one is he or she more likely to choose now?

We have learned through experience to identify properties and establish selection criteria for apples and oranges and in fact we use that experience to make tradeoffs among the properties and reach a decision. We choose the apple or orange that yields, according to our preferences, the greater value over all the various attributes.

The Analytic Hierarchy Process (AHP) is an approach to decision making. It is designed to cope with both the rational and the intuitive to select the best of a number of alternatives evaluated with respect to several criteria. In this process, the decision maker carries out only simple pairwise comparison judgments which are then used to develop overall priorities for ranking the alternatives. The AHP allows for inconsistency in the judgments and provides a means to improve consistency.

The simplest form used to structure a decision problem is a hierarchy of three levels: the goal of the decision at the top level, followed by a second level of criteria and a third level of alternatives. Hierarchical decomposition of complex systems appears to be a basic device used by the human mind to cope with diversity. One organizes the factors affecting the decision in gradual steps from the general, in the upper levels of the hierarchy, to the particular, in the lower levels. The purpose of the structure is to make it possible to judge the importance of the elements in a given level with respect to some or all of the elements in the adjacent level above. Once the structuring is completed, the AHP is surprisingly simple to apply [5,7].

In this paper we show that there is a real and practical use for eigenvalues and eigenvectors in human affairs. This use is not contrived; we are led to them in a very natural way.

2. How To Structure A Decision Problem

Perhaps the most creative task in making a decision is to decide on what factors to consider in the structure. To a person unfamiliar with the subject there may be some concern about what to include and where to include it.

When constructing hierarchies one must include enough relevant detail to represent the problem as thoroughly as possible, but not too thoroughly to lose sensitivity to change in the elements. Consider the environment surrounding the problem. Identify the issues or attributes that you feel contribute to the solution. Identify the participants associated with the problem. Arranging the goals, attributes, issues, and stakeholders in a hierarchy serves two purposes: It provides an overall view of the complex relationships inherent in the situation; and in the judgement process, the decision maker can assess whether he or she is comparing issues of the same order of magnitude.

The elements being compared should be homogeneous. (See axioms in reference [6]). The hierarchy does not need to be complete, that is, an element in a given level does not have to function as a criterion for <u>all</u> the elements in the level below. Thus a hierarchy can be divided into subhierarchies sharing only a common topmost element. Further, a decision maker can insert or eliminate levels and elements as necessary to clarify the task of setting priorities or to sharpen the focus on one or more parts of the system. Elements that are of less immediate interest can be represented in general terms at the higher levels of the hierarchy and elements critical to the problem at hand can be developed in greater depth and specificity. The task of setting priorities requires that the criteria, the subcriteria, the properties or features of the alternatives being compared, and the alternatives themselves are gradually layered in the hierarchy so that the elements in each level are comparable among themselves in relation to the elements of the next higher level.

Finally, after judgments have been made on the impact of all the elements, and priorities have been computed for the hierarchy as a whole, sometimes, and with care, the less important elements can be dropped from further consideration because of their relatively small impact on the overall objective. The priorities can then be recomputed throughout, either with or without, changing the remaining judgments.

3. Paired Comparisons

When we measure something with respect to a property, we usually use some known scale for that purpose. If there is no such scale we can derive one using judgments to make paired comparisons. One of the uses of a hierarchy is that it allows us to focus separately on each of several properties essential for making a sound decision. The most effective way to concentrate judgement, is to take a pair of elements and compare them on a single property without concern for other properties or other elements. This is why paired comparisons in combination with the hierarchical structures are so useful in deriving measurement.

Assume that we are given n stones, $A_1,\dots,A_n$, whose weights $w_1,\dots,w_n$, respectively, are known to us. Let us form the matrix of pairwise ratios whose rows give the ratios of the weights of each stone with respect to all others. Thus we have the matrix:

$$A = \begin{array}{c|cccc} & A_1 & A_2 & \cdots & A_n \\ \hline A_1 & w_1/w_1 & w_1/w_2 & \cdots & w_1/w_n \\ A_2 & w_2/w_1 & w_2/w_2 & \cdots & w_2/w_n \\ \vdots & \vdots & \vdots & & \vdots \\ A_n & w_n/w_1 & w_n/w_2 & \cdots & w_n/w_n \end{array} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = n \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix}$$

We have multiplied A on the right by the vector of weights $w = (w_1, w_2, \dots, w_n)^T$. The result of this multiplication is nw, in other words, n is an eigenvalue of A with eigenvector w. Now A has rank one since every row is a constant multiple of the first row. Thus all its eigenvalues except one are zero. The sum of the eigenvalues of a matrix is equal to its trace, the sum of the diagonal elements, and in this case, the trace of A is equal to n. Therefore, n is the largest, or principal, eigenvalue of A.

The solution of Aw = nw, called the principal right eigenvector of A, consists of positive entries and is unique to within a multiplicative constant. To make w unique, we normalize its entries by dividing by their sum. It is clear that if we are given the comparison matrix A, we can recover the scale. In this case the solution is the normalized version of any column of A.

The matrix $A = (a_{ij})$, $a_{ij} = w_i/w_j$, $i,j = 1,\dots,n$, has positive entries everywhere and satisfies the reciprocal property $a_{ji} = 1/a_{ij}$. Any matrix with this property is called a reciprocal matrix. In addition, A is <u>consistent</u> because the following condition is satisfied:

$$a_{jk} = a_{ik}/a_{ij}, \quad i,j = 1,\dots,n \tag{1}$$

We see that the entire matrix can be constructed from a set of n elements which form a chain (or more generally, a spanning tree, in graph-theoretic terminology) across the rows and columns.

In a more general decision-making environment, we cannot give the precise values of the w_i/w_j but only estimates of them. Let us consider estimates of these values given by an expert who may make small errors in judgment. Our problem now becomes

$A w = \lambda_{max} w$ where λ_{max} is the principal eigenvalue of A where A may no longer be consistent but is still reciprocal. The problem now is: to what extent does w reflect the expert's actual opinion? Note that if we obtain w by solving this problem, and then form a matrix with the entries (w_i/w_j), we obtain an approximation of A, which is a consistent matrix.

We now show the interesting, and perhaps surprising result that inconsistency throughout the matrix can be captured by a single number $\lambda_{max} - n$, which measures the deviation of the judgments from the consistent approximation.

Let $a_{ij} = (1+\varepsilon_{ij})\, w_i/w_j$, $\varepsilon_{ij} > -1$, be a perturbation of w_i/w_j, where w is the principal eigenvector of A.

<u>Theorem 1</u>: $\lambda_{max} \geq n$

<u>Proof</u>: Using $a_{ji} = 1/a_{ij}$, and $A w = \lambda_{max} w$, we have

$$\lambda_{max} - n = \frac{1}{n} \sum_{1 \leq i < j \leq n} \frac{\varepsilon_{ij}^2}{1 + \varepsilon_{ij}} \geq 0 \tag{2}$$

<u>Theorem 2</u>: A is consistent if and only if $\lambda_{max} = n$.

<u>Proof</u>: If A is consistent, then because of (1), each row of A is a constant multiple of a given row. This implies that the rank of

A is one, and all but one of its eigenvalues λ_i, $i = 1,\ldots,n$, are zero. However, it follows from our earlier argument that, $\sum_{i=1}^{n} \lambda_i$ = Trace (A) = n. Therefore $\lambda_{max} = n$.

Conversely, $\lambda_{max} = n$, implies $\varepsilon_{ij} = 0$, and $a_{ij} = w_i/w_j$.

For the consistency index (C.I.), we adopt the value $(\lambda_{max} - n)/(n - 1)$. It is the negative average of the other roots of the Characteristic polynomial of A. This value is compared with the same index obtained as an average over a large number of reciprocal matrices of the same order whose entries are random. If the consistency ratio (C.R.) of C.I. to that from random matrices is significantly small (e.g., 10% or less), we accept the estimate of w. Otherwise, we attempt to improve consistency.

The reader may have heard of the experimental findings of the pschologist George Miller in the 1950's [4]. He found that in general, people (such as chess experts) could deal with information involving simultaneously only a few facts, seven plus or minus two, he wrote. With more, they become confused and cannot handle the information. This is in harmony with the stability of the principal eigenvalue to small perturbations when n is small [6], and its central role in the measurment of consistency.

4. Two Examples

The AHP is used with two types of measurement, relative and absolute. In both, paired comparisons are performed to derive priorities for criteria with respect to the goal. In relative measurement, paired comparisons are also performed on the alternatives in the lowest level of the hierarchy with respect to each criterion. In absolute measurement, the level above the alternatives consists of intensities or grades which are refinements of the criteria or subcriteria governing the alternatives. One must be able to compare the grades themselves under each criterion, by answering questions such as: how much better is an excellent student than a very good student, and so on. The alternatives are simply rated according to these grades and as a result of the weighting process receive their overall ranks. This will become clear in the second example below.

A) Relative Measurement: Choosing the Best House to Buy

When advising a family of average income to buy a house, the family identified eight criteria which they thought they had to look for in a house. These criteria fall into three categories: economic, geographic and physical. Although one may have begun by examining the relative importance of these clusters, the family felt they wanted to prioritize the relative importance of all the criteria without working with clusters. The problem was to decide which of three candidate houses to choose. The first step is the structuring of the problem as a hierarchy.

In the first (or top) level is the overall goal of "Satisfaction with House." In the second level are the eight criteria which contribute to the goal, and the third (or bottom) level are the three candidate houses which are to be evaluated in terms of the criteria in the second level. The definitions of the criteria and the pictorial representation of the hierarchy follow.

The criteria important to the individual family were:

(1) Size of House: Storage space; size of rooms, number of rooms; total area of house.

(2) Location to Bus Lines: Convenient, close bus service.

(3) Neighborhood: Little traffic, secure, nice view, low taxes, good condition of neighborhood.

(4) Age of House: Self-explanatory.

(5) Yard Space: Includes front, back and side, and space from neighbors.

(6) Modern Facilities: Dishwashers; garbage disposals; air conditioning; alarm system; and other such items possessed by a house.

(7) General Condition: Repairs needed; walls; carpet; drapes; cleanliness; wiring.

(8) Financing Available: Assumable mortgage; seller financing available, or bank financing.

The next step is the Comparative Judgment step. Arrange the elements in the second level into a matrix and elicit judgments from the people who have the problem about the relative importance of the elements with respect to the overall goal, Satisfaction with House. The scale to use in making the judgments is given in Table 1. This scale has been validated for effectiveness, not only in many applications by a number of people [3], but also through theoretical comparisons with a large number of other scales.

The questions to ask when comparing two criteria are of the following kind: of the two criteria being compared, which is considered more important by the family buying the house with respect to the overall goal of family satisfaction with the house?

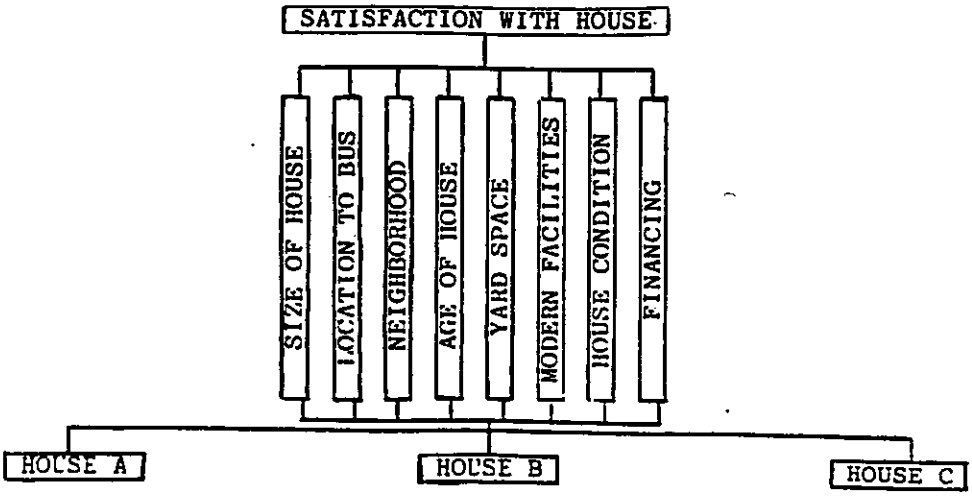

FIGURE 1: Decomposition of the Problem into a Hierarchy

The matrix of pairwise comparisons of the criteria given by the homebuyers in this case is shown in Table 2, along with the resulting vector of priorities. The vector of priorities is the principal eigenvector of the matrix. It gives the relative priority of the criteria measured on a ratio scale.

TABLE 1

THE FUNDAMENTAL SCALE

Intensity of Importance on an Absolute Scale	Definition	Explanation
1	Equal importance.	Two activities contribue equally to the objective.
3	Moderate importance of one over another.	Experience and judgment strongly favor one activity over another.
5	Essential or strong importance.	Experience and judgment strongly favor one activity over another.
7	Very strong importance.	An activity is strongly favored and its dominance demonstrated in practice.
9	Extreme importance.	The evidence favoring one activity over another is of the highest possible order of affirmation.
2,4,6,8	Intermediate values between the two adjacent judgments.	When compromise is needed.
Reciprocals	If activity i has one of the above numbers assigned to it when compared with activity j, then j has the reciprocal value when compared with i.	
Rationals	Ratios arising from the scale.	If consistency were to be forced by obtaining n numerical values to span the matrix.

When the elements being compared are closer together than indicated by the scale, one can use the scale 1.1, 1.2, ..., 1.9. If still finer, one can use the appropriate percentage refinement.

TABLE 2

Pairwise Comparison Matrix for Level 2

	1	2	3	4	5	6	7	8	Priority Vector
1	1	5	3	7	6	6	1/3	1/4	.173
2	1/5	1	1/3	5	3	3	1/5	1/7	.054
3	1/3	3	1	6	3	4	6	1/5	.188
4	1/7	1/5	1/6	1	1/3	1/4	1/7	1/8	.018
5	1/6	1/3	1/3	3	1	1/2	1/5	1/6	.031
6	1/6	1/3	1/4	4	2	1	1/5	1/6	.036
7	3	5	1/6	7	5	5	1	1/2	.167
8	4	7	5	8	6	6	2	1	.333

λ_{max} = 9.669

C.I. = .238

C.R. = .169

In this case financing has the highest priority with 33% of the influence.

In Table 2, instead of naming the criteria, we use the number previously associated with each. Next we move to the pairwise comparisons of the elements in the lowest level. The elements to be compared pairwise are the houses with respect to how much better one is than the other in satisfying each criterion in level 2. Thus there will be eight 3 X 3 matrices of judgments since there are eight elements in level 2, and 3 houses to be pairwise compared for each element. Again, the matrices contain the judgments of the family involved. To understand the judgments, a brief description of the houses follows.

<u>House A</u> - This house is the largest of them all. It is located in a good neighborhood with little traffic and low taxes. Its yard space is comparably larger than houses B and C. However, the general condition is not very good and it needs cleaning and painting. Also, the financing is unsatisfactory because it would have to be bank-financed at high interest.

<u>House B</u> - This house is a little smaller than House A and is not close to a bus route. The neighborhood gives one the feeling of insecurity because of traffic conditions. The yard space is fairly small and the house lacks the basic modern facilities. On the other hand, the general condition is very good. Also, an assumable mortgage is obtainable which means the financing is good with a rather low interest rate.

<u>House C</u> - House C is very small and has few modern facilities. The neighborhood has high taxes, but is in good condition and seems secure. The yard space is bigger than that of House B, but is not comparable to House A's spacious surroundings. The general condition of the house is good and it has a pretty carpet and drapes.

The matrices of comparisons of the houses with respect to the criteria and their local priorities are given in Table 3.

The next step is to establish the composite or global ities of the houses. We lay out the local priorities of the house with respect to each criterion in a matrix and multiply each column of vectors by the priority of the corresponding criterion and add across each row which results in the desired vector of the houses. House A which was the least desirable with respect to financing (the highest priority criterion), contrary to expectation, had the largest priority. It was the house that was bought.

	1 (.173)	2 (.054)	3 (.188)	4 (.018)	5 (.031)	6 (.036)	7 (.167)	8 (.333)	=
A	.754	.233	.754	.333	.674	.747	.200	.072	.396
B	.181	.055	.065	.333	.101	.060	.400	.650	.341
C.	.065	.713	.181	.333	.226	.193	.400	.278	.263

B) Absolute Measurement: Employee Evaluation

Absolute measurement is applied to rank alternatives in terms of ratings, intensities or grades of the criteria. These grades may take the form: excellent, very good, good, average, below average, poor and very poor. After establishing a scale of priorities for the criteria (or subcriteria, if there are some) through paired comparisons, the grades which may be different for

TABLE 3

Pairwise Comparison Matrices for Level 3

Size of House	A	B	C	Priority Vector
A	1	6	8	.754
B	1/6	1	4	.181
C	1/8	1/4	1	.065

λ_{max} = 3.136
C.I. = .068
C.R. = .117

Yard Space	A	B	C	Priority Vector
A	1	5	4	.674
B	1/5	1	1/3	.101
C	1/4	3	1	.226

λ_{max} = 3.086
C.I. = .043
C.R. = .074

Location to Bus	A	B	C	Priority Vector
A	1	7	1/5	.233
B	1/7	1	1/8	.005
C	5	8	1	.713

λ_{max} = 3.247
C.I. = .124
C.R. = .213

Modern Facilities	A	B	C	Priority Vector
A	1	8	6	.747
B	1/8	1	1/5	.060
C	1/6	5	1	.193

λ_{max} = 3.197
C.I. = .099
C.R. = .170

Neighborhood	A	B	C	Priority Vector
A	1	8	6	.745
B	1/8	1	1/4	.065
C	1/6	4	1	.181

λ_{max} = 3.130
C.I. = .068
C.R. = .117

General Condition	A	B	C	Priority Vector
A	1	1/2	1/2	.200
B	2	1	1	.400
C	2	1	1	.400

λ_{max} = 3.000
C.I. = .000
C.R. = .000

Age of House	A	B	C	Priority Vector
A	1	1	1	.333
B	1	1	1	.333
C	1	1	1	.333

λ_{max} = 3.000
C.I. = .000
C.R. = .000

Financing	A	B	C	Priority Vector
A	1	1/7	1/5	.072
B	7	1	3	.650
C	5	1/3	1	.278

λ_{max} = 3.065
C.I. = .032
C.R. = .056

each criterion or subcriterion, are in turn pairwise compared according to their parent criterion. An alternative is evaluated, for each criterion or subcriterion, by identifying the grade which best describes it. Finally, the weighted or global priorities of the grades are added to produce a ratio scale score for the alternative. Absolute measurement needs standards to make it possible to judge whether the alternative is acceptable or not. Absolute measurement is useful in student addmission, faculty tenure and promotion, employee evaluation, and in other areas where there is fairly good agreement on standards which are then used to rate alternatives one at a time.

Let us consider an abbreviated version of the problem of evaluating employee performance. The hierarchy for the evaluation and the priorities derived through paired comparisons are shown below. It is then followed by a rating of each employee for the quality of performance under each criterion and summing the resulting scores to obtain his overall rating. The hierarchy in Figure 2 can be more elaborate, including subcriteria, followed by the intensities for expressing quality.

Goal: Employee Performance Evaluation

Criteria:	Technical (.061)	Maturity (.196)	Writing skills (.043)	Verbal skills (.071)	Timely work (.162)	Potential (personal) (.466)
Intensities:	Excell. (.604)	Very (.731)	Excell. (.733)	Excell. (.750)	Nofollup (.731)	Great (.750)
	Abv.Avg. (.245)	Accep. (.188)	Averag. (.199)	Averag. (.171)	On Time (.188)	Averag. (.171)
	Averag. (.105)	Immat. (.181)	Poor (.068)	Poor (.078)	Remind (.081)	Bel.Av. (.078)
	Bel.Avg. (.046)					

Figure 2 : The Hierarchy of Employee Evaluation

Alternatives:

1) Mr. X	Excell	Very	Average	Excell.	OnTime	Great
2) Ms. Y	Averg.	Very	Average	Averag.	Nofollup	Average
3) Mr. Z	Excell	Immat.	Average	Excell.	Remind	Great

Let us now show how to obtain the total score for Mr. X:

$$.061 \times .604 + .196 \times .731 + .043 \times .199 + .071 \times .750 +$$

$$.162 \times .188 + .466 \times .750 = .623$$

Similarly the scores for Ms. Y and Mr. Z can be shown to be .369 and .478 respectively. It is clear that we can rank any number of candidates along these lines.

Here the vector of priorities of the criteria has been weighted by the vector of relative number of intensities under each criterion and then renormalized. We call this a structural rescaling of the priorities.

5. Theoretical Considerations

If $A = (a_{ij})$, $a_{ij} > 0$, $i,j = 1,\ldots, n$, Perron proved that A has a unique positive eigenvalue λ_{max} (called the principal eigenvalue of A) that is simple and $\lambda_{max} > |\lambda_k|$ for the remaining eigenvalues of A [1,6]. Furthermore, the principal eigenvector $w = (w_1, \ldots, w_n)$ that is a solution of $Aw = \lambda_{max} w$ has $w_i > 0$, $i = 1,\ldots, n$. We can write the norm of the vector w as, $||w|| = e\ w$ where $e = (1,1,\ldots,1)$, and normalize w by dividing it by its norm. For uniqueness when we refer to w we mean its normalized form. Our purpose here is to show how important the principal eigenvector is in determining the rank of the alternatives through dominance walks.

There is a natural way to derive the rank order of a set of alternatives from a pairwise comparison matrix A. The rank order of each alternative is the relative proportion of its dominance over the other alternatives. This is obtained by adding the elements in each row in A and dividing by the total over all the rows. However, A only captures the dominance of one alternative over each other in one step. But an alternative can dominate a second by first dominating a third alternative and then the third dominates the second. Thus, the first alternative dominates the second in two steps. It is known that the result for dominance in two steps is obtained by squaring the pairwise comparison matrix. Similarly, dominance can occur in three steps, four steps and so on, the value of each obtained by raising the matrix to the corresponding power. The rank order of an alternative is the sum of the relative values for dominance in its row, in one step, two steps and so on averaged over the number of steps. The question is whether this average tends to a meaningful limit.

We can think of the alternatives as the nodes of a directed graph. With every directed arc from node i to node j (which need not be distinct), is associated a nonnegative number a_{ij} of the dominance matrix. In graph-theoretic terms this is the intensity of the arc. Define a k-walk to be a sequence of k arcs such that the terminating node of each arc except the last is the source node of the arc which succeeds it. The <u>intensity of a k-walk</u> is the product of the intensities of the arcs in the walk. With these ideas, we can interpret the matrix A^k: the (i,j) entry of

A^k is the sum of the intensities of all k-walks from node i to node j.

Definition: The dominance of an alternative along all walks of length $k \leq m$ is given by

$$\frac{1}{m}\sum_{k=1}^{m} \frac{A^k e}{e^T A^k e} \quad (3)$$

Observe that the entries of $A^k e$ are the row sums of A^k and that $e^T A^k e$ is the sum of all the entries of A.

Theorem 3: The dominance of each alternative along all walks k, as k is given by the solution of the eigenvalue problem $Aw = \lambda_{max} w$.

Proof:

Let

$$s_k = \frac{A^k e}{e^T A^k e} \quad (4)$$

and

$$t_m = \frac{1}{m}\sum_{k=1}^{m} s_k \quad (5)$$

The convergence of the components of t_m to the same limit as the components of s_m is the standard Cesaro summability. Since,

$$s_k = \frac{A^k e}{e^T A^k e} \rightarrow w \text{ as } k \rightarrow \infty \quad (6)$$

where w is the normalized principal right eigenvector of A, we have

$$t_m = \frac{1}{m}\sum_{k=1}^{m} \frac{A^k e}{e^T A^k e} \rightarrow w \text{ as } m \rightarrow \infty \quad (7)$$

The solution is obtained by raising the matrix A to a sufficiently large power then summing over the rows and normalizing to obtain the priority vector $w = (w_1, \ldots, w_n)$. The process is stopped when the difference between components of the priority vector obtained at the kth power and at the (k+1)th power is less than some pre-determined small value.

In reference [6] we gave at least five different ways of deriving the priorities from the matrix of paired comparisons. Besides the eigenvector solution, these include the direct row sum average, the normalized column average, and methods which minimize the sum of the errors of the differences between the judgments and their derived values such as the methods of least squares and logarithmic least squares. We pointed out that the logarithmic least squares solution coincides with the principal right eigenvector solution for matrices of order n = 3, which is the first value of n in which inconsistency is possible and left and right eigenvectors are reciprocals of each other which is not always the case for larger values of n. Since the appearance of the Analytic Hierarchy Process in the literature, a number of additional methods have been proposed [8,9]. All methods yield the same answer when the matrix is consistent. The combined use of a measure of inconsistency which can be derived in terms of both left and right eigenvectors, along with the right eigenvector solution which captures the dominance expressed in the judgments, is an effective way to look at the problem. We argue that so long as inconsistency is tolerated, dominance is the basic theoretical concept for deriving a scale and not some other mathematically attractive ideas farther removed from the reality of the thinking of the people making the decision.

The software package Expert Choice, useful in teaching and in real applications, incorporates relative and absolute measurement along with structural rescaling, group judgments, sensitivity analysis and dependence among the decision alternatives [2]. The reader interested in pursuing the subject further should consult Mathematical Modelling, Volume 9, Number 3-5, 1987.

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