An Update on Combinatorial Method for Aggregation of Expert Judgments in AHP

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Presentation outline

1. Priority calculation problem statement
2. Alternative methods for deriving priorities from judgments in AHP
3. Overview of combinatorial method and how it works
4. Combinatorial method using weighted geometric mean
5. Calculation of spanning tree ratings in combinatorial method
6. Conceptual advantages of the modified method
7. Empirical results
8. Limitations and further research

Problem statement

What is given:

A_l, $l \in [1..m]$ – expert PCMs, $A=\{a_{ij}\}$, $i,j \in [1..n]$

PCMs properties:

1) reciprocally-symmetrical;
2) multiplicative (or additive);
3) in the general case - incomplete;
4) every single element a_{ij} is obtained in some specific estimation scale;

c_l, $l \in [1..m]$ – relative competence of experts in the group ($\Sigma c_l=1$).

We should find:

The resulting object weight (priority) vector w_k, $k \in [1..n]$ ($\Sigma w_k=1$).

$$\begin{pmatrix} 1 & a_{12} & \dots & \dots & a_{1n} \\ & 1 & \dots & a_{ij} & \dots \\ & & 1 & \dots & \dots \\ & & & 1 & a_{n-1,n} \\ & & & & 1 \end{pmatrix}$$

$$\{(w_1, \dots, w_n);\ \sum_{k=1}^{n} w_k = 1\}$$

Priority calculation methods

- Eigenvector method
- Geometric mean (GM)
- Arithmetic mean
- (Logarithmic) least squares (LLS)
- Combinatorial method (a.k.a. enumeration of all spanning trees (EAST)) – ***ordinary*** or ***modified***
- Aggregation of individual judgments and/or priorities (under group estimation) (Saaty & Peniwati 2007)
- Others

Equivalence of priority calculation methods

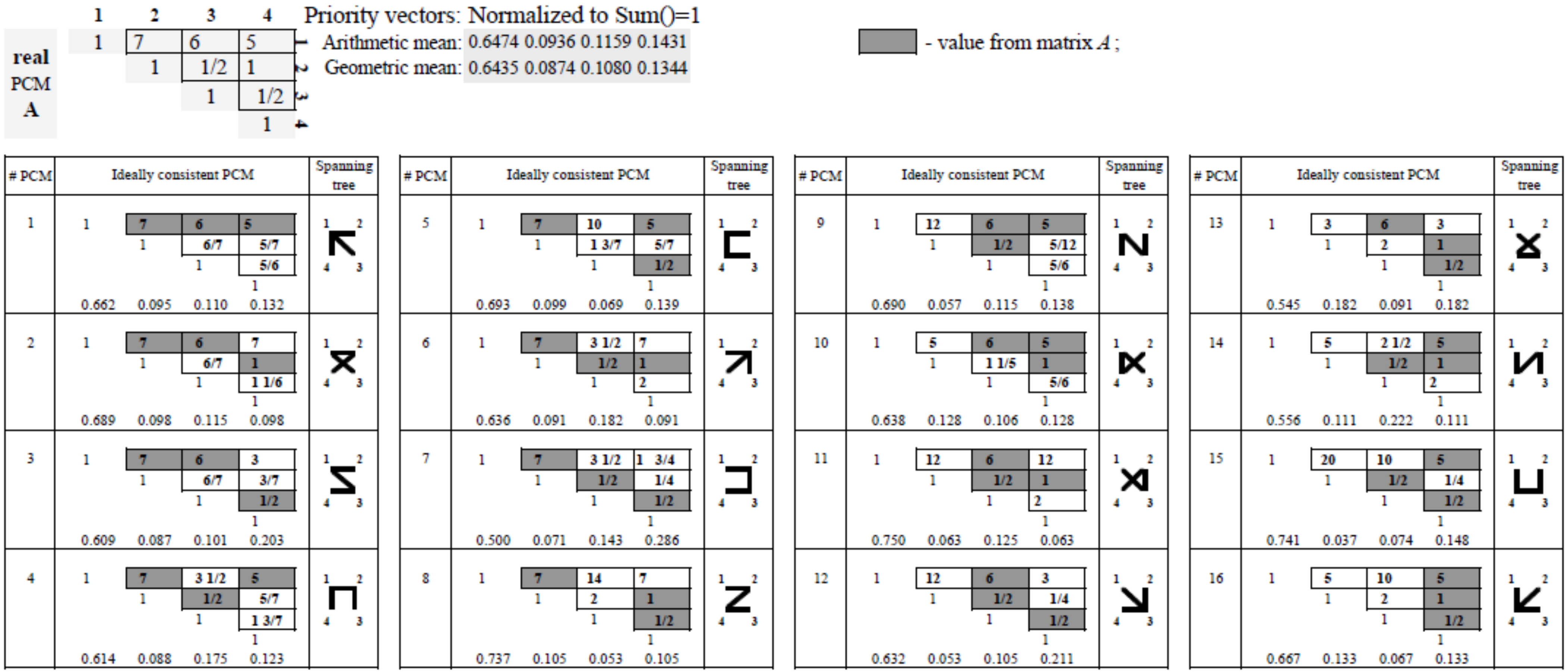
Row GM ⟷ ***Combi (EAST)*** ⟷ ***LLS***

Lundy, M., Siraj, S., & Greco, S. (2017). The Mathematical Equivalence of the “Spanning Tree” and Row Geometric Mean Preference Vectors and its Implications for Preference Analysis. *European Journal of Operational Research* 257(1), 197-208.

Bozoki, S. & Tsyganok, V. (2019). The (logarithmic) least squares optimality of the arithmetic (geometric) mean of weight vectors calculated from all spanning trees for incomplete additive (multiplicative) pairwise comparison matrices. *International Journal of General Systems* 48(4), 362-381.

So, does it make sense to keep using the combinatorial method when less computationally complex equivalents are available?

How combinatorial method works



real PCM **A**

	1	2	3	4
1	1	7	6	5
2		1	1/2	1
3			1	1/2
4				1

Priority vectors: Normalized to Sum()=1

Arithmetic mean: 0.6474 0.0936 0.1159 0.1431

Geometric mean: 0.6435 0.0874 0.1080 0.1344

▨ - value from matrix *A* ;

# PCM	Ideally consistent PCM	Spanning tree
1	1 7 6 5 1 6/7 5/7 1 5/6 1 0.662 0.095 0.110 0.132	1 2 4 3
2	1 7 6 7 1 6/7 1 1 1 1/6 1 0.689 0.098 0.115 0.098	1 2 4 3
3	1 7 6 3 1 6/7 3/7 1 1/2 1 0.609 0.087 0.101 0.203	1 2 4 3
4	1 7 3 1/2 5 1 1/2 5/7 1 1 3/7 1 0.614 0.088 0.175 0.123	1 2 4 3
5	1 7 10 5 1 1 3/7 5/7 1 1/2 1 0.693 0.099 0.069 0.139	1 2 4 3
6	1 7 3 1/2 7 1 1/2 1 1 2 1 0.636 0.091 0.182 0.091	1 2 4 3
7	1 7 3 1/2 1 3/4 1 1/2 1/4 1 1/2 1 0.500 0.071 0.143 0.286	1 2 4 3
8	1 7 14 7 1 2 1 1 1/2 1 0.737 0.105 0.053 0.105	1 2 4 3
9	1 12 6 5 1 1/2 5/12 1 5/6 1 0.690 0.057 0.115 0.138	1 2 4 3
10	1 5 6 5 1 1 1/5 1 1 5/6 1 0.638 0.128 0.106 0.128	1 2 4 3
11	1 12 6 12 1 1/2 1 1 2 1 0.750 0.063 0.125 0.063	1 2 4 3
12	1 12 6 3 1 1/2 1/4 1 1/2 1 0.632 0.053 0.105 0.211	1 2 4 3
13	1 3 6 3 1 2 1 1 1/2 1 0.545 0.182 0.091 0.182	1 2 4 3
14	1 5 2 1/2 5 1 1/2 1 1 2 1 0.556 0.111 0.222 0.111	1 2 4 3
15	1 20 10 5 1 1/2 1/4 1 1/2 1 0.741 0.037 0.074 0.148	1 2 4 3
16	1 5 10 5 1 2 1 1 1/2 1 0.667 0.133 0.067 0.133	1 2 4 3

Ordinary and modified combinatorial method

- ***Ordinary method***: aggregation using simple geometric mean*

$$w_j^{aggregate} = (\prod_{q=1}^{T} w_j^q)^{\frac{1}{T}} \; ; j = 1..n; T \in [1..mn^{n-2}]$$

*total number of "trees" *T* is calculated based on Caley's formula

- ***Modified method***: aggregation using "weighted" geometric mean

$$w_j^{aggregate} = \prod_{k,l=1}^{m} (\prod_{q^k=1}^{T_k} (w_j^{(kq_k l)})^{\frac{R_{kq_k l}}{\sum_{u,p,v} R_{upv}}}) \, ; j = 1..n$$

Ratings of spanning trees

Ratings should reflect 1) consistency, 2) compatibility (in case of group estimation), 3) detail, and 4) completeness of expert judgments.

- Additive case

$$R_{kql} = \frac{c_k c_l s^{kq} s^l}{\ln(\sum_{u,v} \left| a_{uv}^{kq} - a_{uv}^{l} \right| + e)}$$

- Multiplicative case

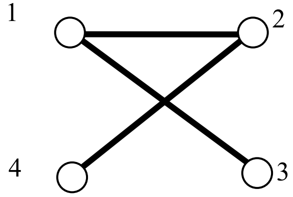
$$R_{kql} = \frac{c_k c_l s^{kq} s^l}{\ln(\prod_{u,v} \max(\frac{a_{uv}^{kq}}{a_{uv}^{l}}; \frac{a_{uv}^{l}}{a_{uv}^{kq}}) + e - 1)}$$

Degree of detail of expert judgment set

- Hartley’s formula for signal transmission applied to estimation scales

$$S_N = I = \log_2 N$$

- For a basic pair-wise comparison set (tree) of *n*-1 elements (nodes)



$$s^{kq} = (\prod_{u=1}^{n-1} \log_2 N_u^{(kq)})^{\frac{1}{n-1}}$$

- For a PCM of dimensionality *n* $s^k = (\prod_{\substack{u,v=1\\v>u}}^{n} \log_2 N_{uv}^{(k)})^{\frac{2}{n(n-1)}}$

Example (3 experts compare 4 objects)

Number of grades in the scales, selected by experts for pair-wise comparisons

	E_1				E_2				E_3			
	A_1	A_2	A_3	A_4	A_1	A_2	A_3	A_4	A_1	A_2	A_3	A_4
A_1	1	9	8	7	1	3	4	5	1	9	9	8
A_2		1	6	5		1	6	7		1	3	9
A_3			1	4			1	8			1	7
A_4				1				1				1

Numbers of specific grades in the pair-wise comparison scales, selected by the experts

	E_1				E_2				E_3			
	A_1	A_2	A_3	A_4	A_1	A_2	A_3	A_4	A_1	A_2	A_3	A_4
A_1	1	2	4	7	1	3	4	5	1	2	4	8
A_2		1	2	4		1	2	3		1	2	5
A_3			1	2			1	2			1	3
A_4				1				1				1

Example (continued)

Values of pair-wise comparisons, brought to the unified scale (Tsyganok et al, 2015)

	E_1				E_2				E_3			
	A_1	A_2	A_3	A_4	A_1	A_2	A_3	A_4	A_1	A_2	A_3	A_4
A_1	1	2	4 1/3	8 5/6	1	7 1/2	8 1/6	8 1/2	1	2	4	9
A_2	1/2	1	2 2/7	6 1/2	1/7	1	2 2/7	3 1/2	1/2	1	3 1/2	5
A_3	2/9	3/7	1	2 5/6	1/8	3/7	1	2	1/4	2/7	1	3 1/2
A_4	1/9	1/6	1/3	1	1/8	2/7	1/2	1	1/9	1/5	2/7	1

Resulting priority vector (w_1, w_2, w_3, w_4) calculated using:

1. ***Modified*** method (0.563734299; 0.263382041; 0.120820159; 0.052063501)
2. ***Ordinary*** method (0.590174795; 0.243658012; 0.114086692; 0.052080501)

Results are noticeably different, so the methods are not equivalent!

Conceptual advantages of the method

OK, the modified method is different from other methods, but is it better in some way?

1) It allows to take the quality of expert information into account prior to aggregation and "award" more compatible, consistent, complete, and detailed judgments by assigning greater weights (ratings) to them.

2) In addition to CR and CI used in AHP it uses spectral approach, allowing to organize step-by-step feedback with experts, i.e. request them to reconsider the most inconsistent (incompatible) judgments (Olenko & Tsyganok 2016).

3) The method is universal, i.e. suitable for additive/multiplicative estimates, individual/group judgments, complete/incomplete PCM.

Efficiency of the method: quantitative aspect

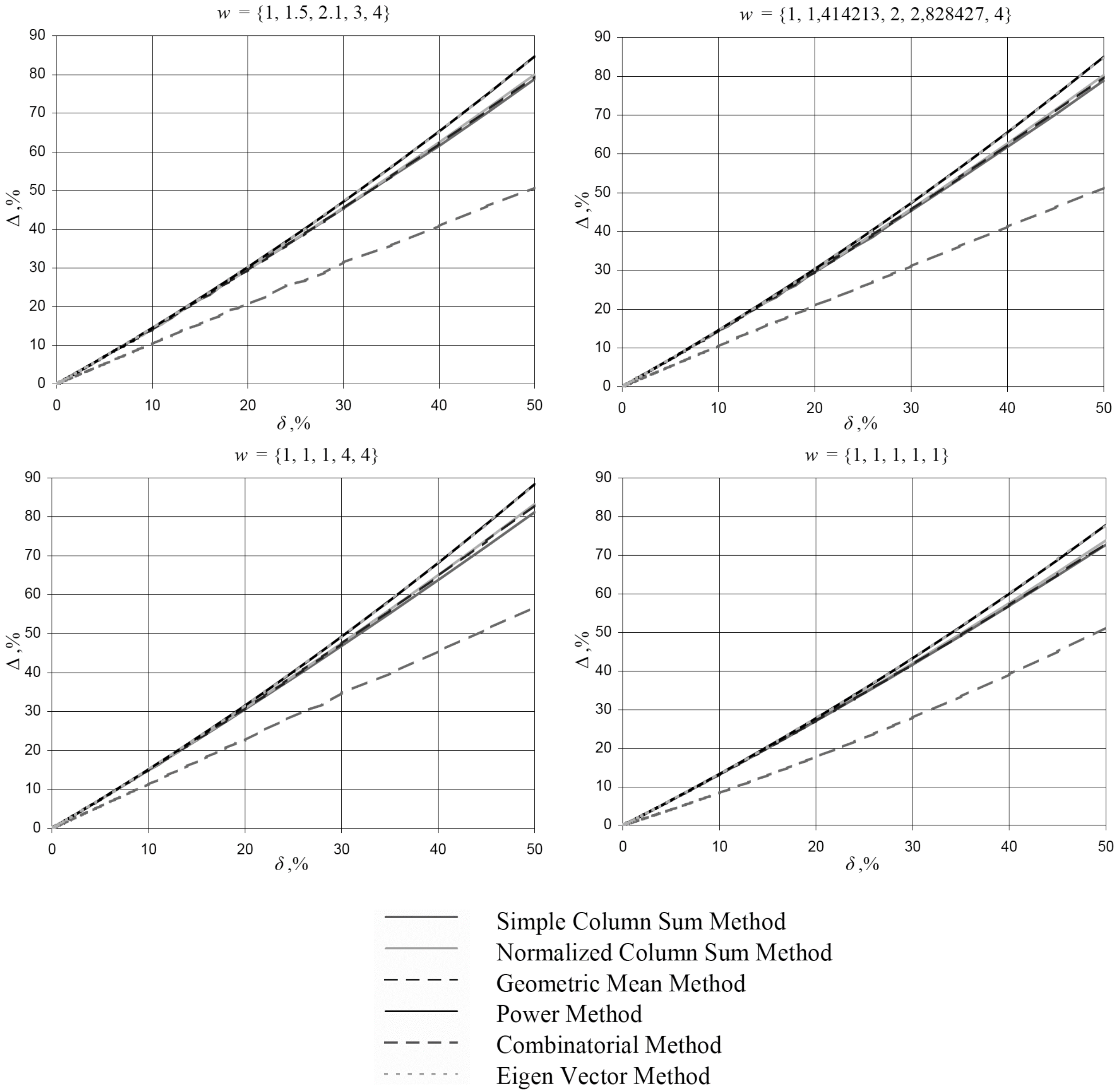
- ***“Accuracy”*** of expert estimation methods is problematic to evaluate and compare, because experts themselves are being inaccurate, even when they evaluate “model” objects according to “tangible” criteria (such as figure squares). They introduce an unnecessary and “uncontrollable” degree of freedom into the process (Kadenko & Tsyganok, 2019).
- However, we can compare the methods in terms of ***sensitivity*** using simulation of the whole expert session process (including estimation). One of the ways to compare the sensitivity of priority calculation methods using simulation (Tsyganok, 2010) is as follows.

1. generate model priority values $\left\{w_j; j = 1..n\right\}$;
2. build a consistent PCM *A* based on these values $\{a_{ij} = \frac{w_j}{w_i}; i, j = 1..n\}$;
3. perturb this PCM $a'_{ij} = a_{ij} \pm a_{ij} \cdot \delta/100\%$
4. calculate priorities based on the perturbed PCM *A’*
5. calculate the deviations of resulting priorities from initial ones $\Delta = \max_i \left|\frac{w'_i - w_i}{w_i}\right| \times 100\%$
6. compare maximum priority deviations obtained using different methods

Simulation results

<u>Examples of priority ratios:</u>

- Equal priorities
- Priorities located at the opposite ends of a numeric interval
- Arithmetic progression
- Geometric progression
- Others



Limitations

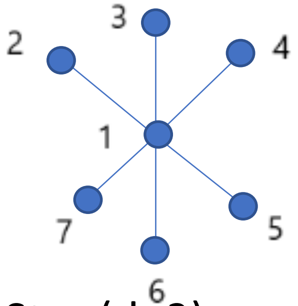
So, is modified method *always* more stable to perturbations of initial data than the ordinary method? Can this be proved analytically?

Empirical comparisons of the two versions of the method ***do not*** allow us to state that the modified method is more accurate for **all** test sets of judgments

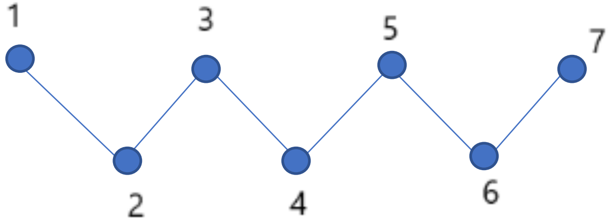
1. Genetic algorithm (GA) we use to calculate the deviations of priorities might, by definition, omit (“skip”) significant results and lead us to local extremes of the fitness function. (*In terms of the GA, the “individuals” are the perturbed PCM with given perturbation (relative “error”) value* δ*. “Fitness function” is the maximum relative deviation of resulting priorities from the true values of object weights* Δ).
2. In addition to perturbation δ, sensitivity of combinatorial aggregation method depends on diameters of specific spanning tree graphs (as shown below).

Limitations (continued)

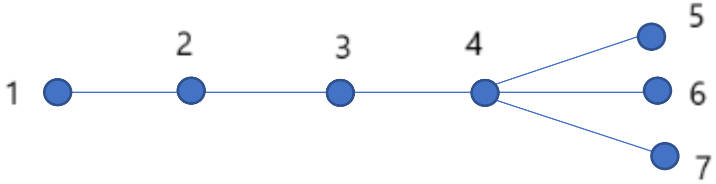
- Examples of non-isomorphic spanning trees with different diameters (*n*=7)



Star (d=2)



Path (d=(n-1)=6)



“Broom” (d=4)

- If elements of initial ICPCM A are perturbed by noise δ, and then some ICPCM A^* is reconstructed based on a spanning tree of diameter $k \in [2, .., (n-1)]$, then the most “deviated” element of A^* is $a_{ij}^* = a_{ij}\,(1 \pm \delta)^{k_1}/(1 \mp \delta)^{k_2}$ where $k_1 + k_2 = k$. Under small δ, $a_{ij}^* \approx a_{ij}\,(1 \pm k\delta)$.

Spanning trees of larger diameter “accumulate” larger estimation errors, making it more difficult to analytically compare ordinary and modified combinatorial methods.

Conclusions and further research

In spite of recently obtained results of (Lundy et al, 2017), (Bozoki & Tsyganok, 2019), it still makes sense to use and further improve the modified combinatorial method for aggregation of expert judgments, because it has certain conceptual advantages over the ordinary method.

Future research on the subject will be dedicated to:

1) Analytical studies and comparison of the ordinary and modified methods’ sensitivity to perturbations of initial data, based on graph theory.
2) Further modifications of the method, possibly, taking into account the diameter of spanning trees.

Key references

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Thank you for attention!

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