EVALUATION OF EIGEN-VALUE METHOD AND GEOMETRIC MEAN BY BRADLEY-TERRY MODEL IN BINARY AHP

Kazutomo Nishizawa
Department of Mathematical Engineering,
College of Industrial Technology, Nihon University
1-2-1 Izumi-cho, Narashino, Chiba 275-8575, Japan
k7nisiza@cit.nihon-u.ac.jp

Keywords: Binary AHP, Bradley-Terry Model, Eigen-Value Method, Geometric Mean

Summary: *AHP (Analytic Hierarchy Process) is a useful tool for decision makers and widely use in the various fields. Typical methods to have weights are Eigen-Value Method and Geometric Mean. Unfortunately we don't know which method is better. In usual evaluation we artificially add error to perfect consistent comparisons and have weights by each method and compare with perfect consistent weights. In general we chose error as normal random numbers. In this case we have better results by Geometric Mean because of Gauss-Markof theorem. In this study we consider another kind of error model, Bradley-Terry Model. This model was proposed by Bradley and Terry and is mainly used for evaluations of sports games. On the other hand binary AHP is also used for sports field. In this paper we evaluate Eigen-Value Method and Geometric Mean by Bradley-Terry Model in binary AHP, through examples and simulation.*

1. Introduction

AHP (Saaty, 1980) is a useful tool for decision makers and widely use in the various fields. Typical methods to have weights are Eigen-Value Method (EV) and Geometric Mean (GM). Unfortunately we don't know which method is better.

In a perfect consistent comparison of n alternatives, a_{ij}, the element of comparison matrix A, is represented by ratio of the alternative's weights, w_i (i=1 to n), that is for any alternatives i and j we have

$$a_{ij} = w_i / w_j . \quad (1)$$

In actual pairwise comparison, caused by overestimate or underestimate, a_{ij} is include error e_{ij}, as follows.

$$a_{ij} = (w_i / w_j)e_{ij} \quad (2)$$

However, it is not easy to estimate e_{ij}. By using the logarithmic scale, eq.(2) is represented by the following equation.

$$\log a_{ij} = \log w_i - \log w_j + \log e_{ij} \quad (3)$$

In experimental simulations, firstly we artificially add e_{ij} to perfect consistent comparisons and construct comparison matrix. Next we have weights by the proposed method and compare with perfect consistent weights. In general we chose log e_{ij} in eq.(3) as normal random numbers of $N(0, \sigma^2)$, where σ is the standard deviation. In this model we have better results by GM because of Gauss-Markof theorem.

So we consider another kind of error model, Bradley-Terry Model (BT model)(Bradley and Terry, 1952), and is used for evaluation of sports games. And, binary AHP is also used for sports field and evaluate the strength of the team (Takahashi, 1990; Nishizawa, 1995).

In this paper we evaluate EV and GM by BT model in binary AHP, through examples and simulation. In section 2 and 3, we briefly explain BT model and binary AHP, and evaluation method. In section 4, we evaluate EV and GM through the baseball games as the practical example. Furthermore, in section 5, simulation is carried out. Finally in section 6, we conclude our investigation.

2. Bradley-Terry Model and Binary AHP

In BT model for the field of sports, the suppose strength of team i to be w_i (i=1 to n), then we assume that the probability p_{ij} of team i to defeat team j, is

$$p_{ij} = w_i / (w_i+w_j). \quad (4)$$

Using P=[p_{ij}] and the uniform random number U on [0,1], we can construct comparison matrix A. For each i and j where i>j, if p_{ij} < (>) U then team i is to defeat (lose to) team j, and let $a_{ij}=\theta(1/\theta)$ and $a_{ji}=1/\theta(\theta)$, where θ is a parameter and θ>1. Of cause $a_{ii}=1$.

3. Evaluation

Let W_{EV} (W_{GM}) be the solution from the given comparison matrix A by EV (GM). Estimating the strength of team i by the i-th element w_{EVi} (w_{GMi}) of W_{EV} (W_{GM}), we have the following Residual Sum of Squares (RSS).

$$RSS_{EV}= \Sigma (w_i - w_{EVi})^2 ,$$

$$RSS_{GM}= \Sigma (w_i - w_{GMi})^2 \qquad (i=1 \text{ to } n) \quad (5)$$

4. Example

To explain BT model and binary AHP, we consider the baseball games as the practical example. We have the winning rate of 135 matches of each six teams, as shown below.

[0.585000 0.556000 0.541000 0.489000 0.444000 0.385000]

Normalizing these with sum of elements equal to 1, we have the following weights.

$$W =[\ 0.195000\ \ 0.185333\ \ 0.180333\ \ 0.163000\ \ 0.148000\ \ 0.128333\ ] \quad (6)$$

Then we estimate the team strength by BT model in binary AHP and compare with W.
Based on eq.(4) and eq.(6) we have BT matrix P, as shown below.

$$P=\begin{bmatrix} 0.500000 & 0.512708 & 0.519538 & 0.544693 & 0.568513 & 0.603093 \\ 0.487292 & 0.500000 & 0.506837 & 0.532057 & 0.556000 & 0.590861 \\ 0.480462 & 0.493163 & 0.500000 & 0.525243 & 0.549239 & 0.584233 \\ 0.455307 & 0.467943 & 0.474757 & 0.500000 & 0.524116 & 0.559497 \\ 0.431487 & 0.444000 & 0.450761 & 0.475884 & 0.500000 & 0.535585 \\ 0.396907 & 0.409139 & 0.415767 & 0.440503 & 0.464415 & 0.500000 \end{bmatrix} \quad (7)$$

Using the elements p_{ij} of P and ${}_6C_2$=15 uniform random numbers we construct binary comparison matrix A_1.

$$A_1 = \begin{bmatrix} 1 & \theta & \theta & 1/\theta & \theta & 1/\theta \\ 1/\theta & 1 & \theta & \theta & \theta & \theta \\ 1/\theta & 1/\theta & 1 & 1/\theta & \theta & \theta \\ \theta & 1/\theta & \theta & 1 & \theta & \theta \\ 1/\theta & 1/\theta & 1/\theta & 1/\theta & 1 & 1/\theta \\ \theta & 1/\theta & 1/\theta & 1/\theta & \theta & 1 \end{bmatrix} \quad (8)$$

Then we have W_{EV} by EV and W_{GM} by GM, from $A_1$where θ=2, as shown below.

W_{EV}=[0.190147 0.224694 0.137701 0.217904 0.081545 0.148009]

W_{GM}=[0.178341 0.224695 0.141549 0.224695 0.089170 0.141549]

From eq.(5), we have RSS_{EV}= 0.011208 and RSS_{GM} =0.010773. Repeating above process, we have A_k (k=1 to 30000), and we obtain the mean values, RSS_{EV}= 0.014633 and RSS_{GM} =0.013470. As a result, in this case, GM is better than EV. However RSS seems to depend on the value of θ.

5.Simulation

In order to confirm the influence of the value of θ, we carried out the following simulation. For n=5, 10, 20, by similar procedure of above example, we estimate the weights by EV and GM for θ= 2, 3,…, 20 and have each RSS. Further, we consider two kind of W. One is an equal interval weight, w_i=i (i=1 to n), and the other is a random weight by uniform random numbers.

Firstly we suppose W be an equal interval weight. For example n=5, w_i as follows.

[1 2 3 4 5]

Normalizing these with sum of w_i equal to 1, we have the following weights W.

W=[0.066666 0.133333 0.200000 0.266666 0.333333]

From simulation for equal interval weights, we have RSS_{EV} and RSS_{GM} for various value of θ.
The results are shown in Table 1.

The graphical representations of the contents of Table 1 are shown in Fig.1 to Fig.3, for n=5, 10, 20, respectively. In each figure, the vertical axis is a value of RSS and the horizontal axis is a value of θ.

Table 1: RSS for Equal Interval Weights

	n=5		n=10		n=20	
θ	EV	GM	EV	MG	EV	GM
2	0.020074	0.019274	0.012346	0.011681	0.006762	0.006437
3	0.012342	0.010126	0.008204	0.006121	0.004542	0.003399
4	0.009017	0.006024	0.006647	0.003522	0.003713	0.001935
5	0.007126	0.003599	0.005858	0.002036	0.003310	0.001111
6	0.006409	0.002539	0.005453	0.001189	0.003101	0.000640
7	0.005456	0.001505	0.005156	0.000673	0.002969	0.000346
8	0.005170	0.001049	0.004983	0.000352	0.002881	0.000169
9	0.004543	0.000587	0.004824	0.000134	0.002829	0.000070
10	0.004170	0.000323	0.004771	0.000060	0.002794	0.000021
11	0.004097	0.000251	0.004696	0.000022	0.002764	0.000008
12	0.003938	0.000177	0.004686	0.000016	0.002736	0.000018
13	0.003855	0.000153	0.004654	0.000044	0.002709	0.000048
14	0.003478	0.000049	0.004543	0.000113	0.002716	0.000085
15	0.003401	0.000084	0.004617	0.000140	0.002695	0.000135
16	0.003469	0.000078	0.004570	0.000224	0.002688	0.000194
17	0.003317	0.000131	0.004571	0.000318	0.002678	0.000258
18	0.003103	0.000128	0.004453	0.000417	0.002680	0.000319
19	0.003196	0.000165	0.004461	0.000504	0.002659	0.000395
20	0.003254	0.000147	0.004521	0.000577	0.002656	0.000464

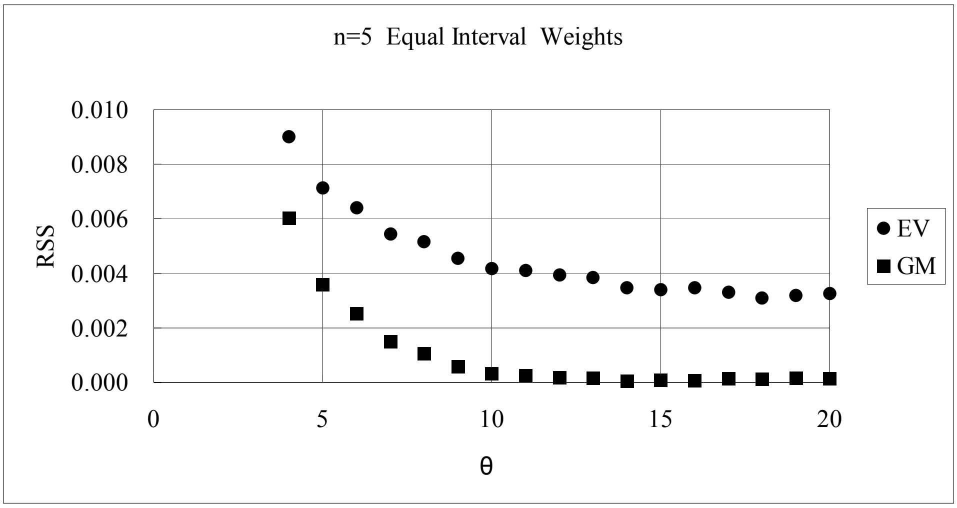

Fig.1 RSS by Equal Interval Weights for n=5

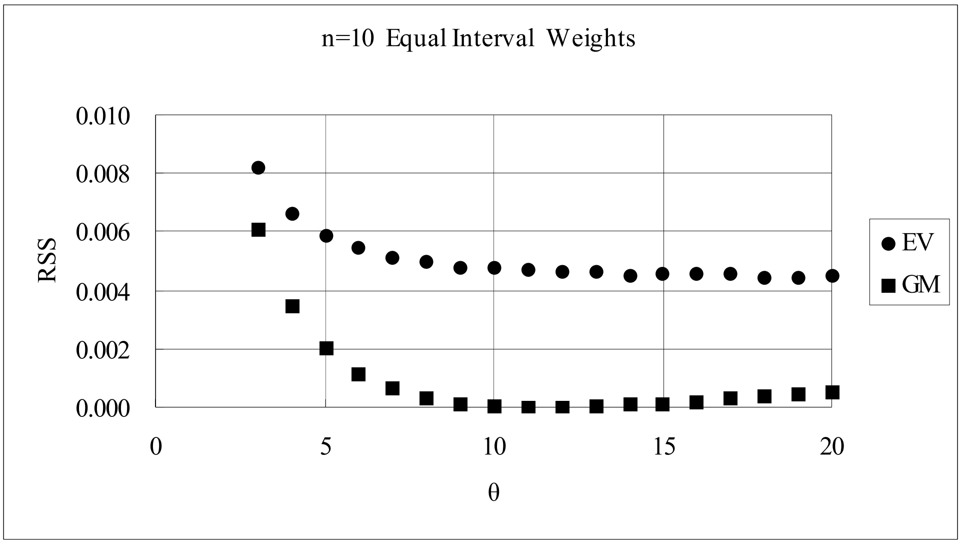

Fig.2 RSS by Equal Interval Weights for n=10

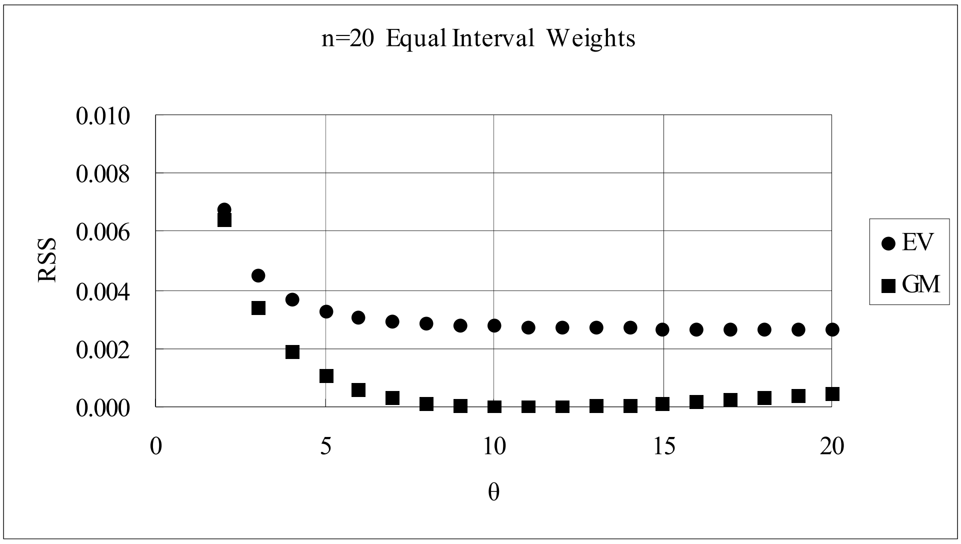

Fig.3 RSS by Equal Interval Weights for n=20

Secondly we suppose W be a random weights. In this case we suppose w_i in [0,1] uniform random numbers. For example n=5, as follows.

[0.146092 0.458449 0.810907 0.179479 0.412427]

Normalizing these with sum of w_i equal to 1, we have the following weights W.

W=[0.072778 0.228385 0.403968 0.089411 0.205458]

From simulation for random weights, we have RSS_{EV} and RSS_{GM} for various value of θ.
The results are shown in Table 2.

Table 2: RSS for Random Weights

	n=5		n=10		n=20	
θ	EV	GM	EV	MG	EV	GM
2	0.030028	0.028870	0.020971	0.020376	0.005719	0.005522
3	0.018323	0.015312	0.013736	0.011429	0.003779	0.002893
4	0.013285	0.009066	0.010817	0.007104	0.003065	0.001635
5	0.010766	0.005918	0.009303	0.004575	0.002737	0.000943
6	0.008869	0.003665	0.008582	0.003111	0.002555	0.000533
7	0.007861	0.002493	0.007958	0.002041	0.002449	0.000290
8	0.007101	0.001689	0.007591	0.001392	0.002379	0.000144
9	0.006356	0.001033	0.007492	0.000988	0.002318	0.000055
10	0.005941	0.000677	0.007210	0.000588	0.002296	0.000020
11	0.005659	0.000482	0.007083	0.000394	0.002278	0.000012
12	0.005208	0.000246	0.006975	0.000223	0.002242	0.000026
13	0.005152	0.000230	0.006909	0.000121	0.002214	0.000060
14	0.004949	0.000131	0.006871	0.000074	0.002215	0.000100
15	0.004688	0.000062	0.006752	0.000034	0.002210	0.000140
16	0.004556	0.000081	0.006755	0.000030	0.002200	0.000195
17	0.004476	0.000116	0.006722	0.000040	0.002194	0.000262
18	0.004276	0.000116	0.006660	0.000070	0.002190	0.000326
19	0.004293	0.000171	0.006694	0.000098	0.002184	0.000390
20	0.004174	0.000186	0.006664	0.000137	0.002179	0.000455

The graphical representations of the contents of Table 2 are shown in Fig.4 to Fig.6, for n=5, 10, 20, respectively.

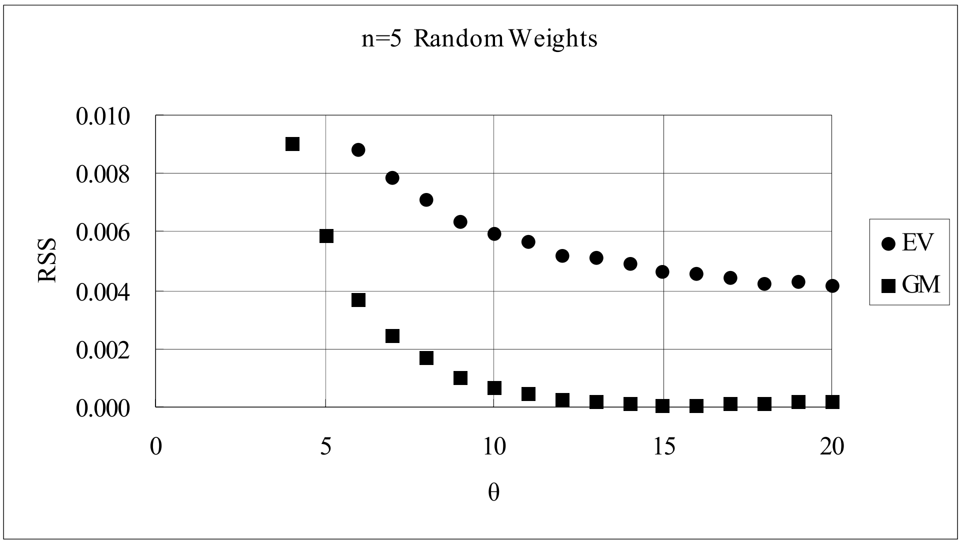

Fig.4 RSS by Random Weights for n=5

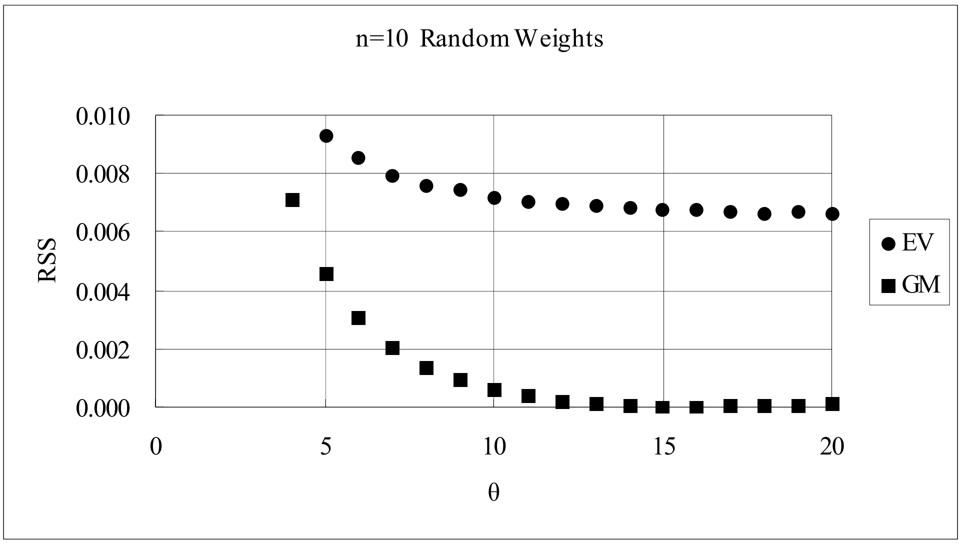

Fig.5 RSS by Random Weights for n=10

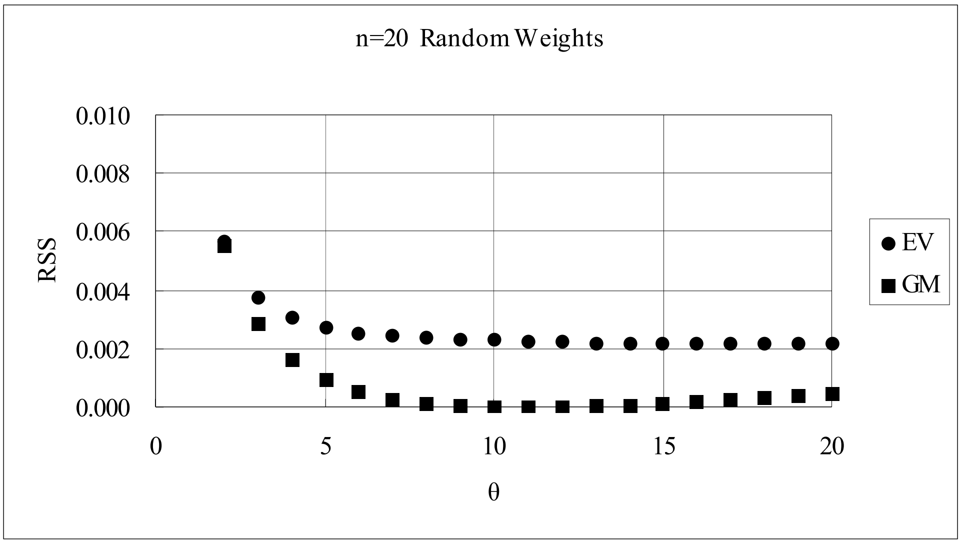

Fig.6 RSS by Random Weights for n=20

6. Conclusion

In this paper we compare EV and GM by BT-model in binary AHP. As a result, based on RSS from simulation, GM is better than EV for various matrix sizes and various value of θ. In binary AHP, we usually use θ =2. But from the results we may have better to use larger value of θ than 2. Furthermore we need to consider the relation between RSS and θ. We believe that the value of θ to minimize the value of RSS is exist. We need to study it in future.

References

Saaty, T. L. (1980) "The Analytic Hierarchy Process", McGrawHill, NewYork.

Bradly, R. A. and Terry, M. E. (1952) "Rank analysis of incomplete block designs. 1. The method of paired comparisons", *Biometrika*, 39, 234-335.

Takahashi, I. (1990) "AHP Applied to Binary and Ternary Comparisons", *Journal of Operations Research Society of Japan*, Vol. 33, No. 3, 199-206.

Nishizawa, K. (1995) "A Consistency Improving Method in Binary AHP", *Journal of Operations Research Society of Japan*, Vol.38, No.1, 21-33.