EVALUATING COMPARISON BETWEEN CONSISTENCY IMPROVING METHOD AND RESURVEY IN AHP

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Summary : *Resurvey is preferable to achieve accuracy and consistency in revised matrix although it takes relatively longer time than consistency improving method. It is proven by experiment that trere are only 46,88% matrices can be revised by consistency improving method, and only 66,67% of them are significantly identic to the result of resurvey.*

1. Introduction

A comparison matrix is said to be acceptable if its consistency ratio is less than 0,1. Then in general case, however it also difficult to obtain such as matrix, especially for the matrix with high order, because of the influence of the limited ability of human thinking and the shortcomings of one to nine scale. There may be a way of dealing with matrices with unacceptable consistency (CR>0,1). That is, returning such matrices to experts to reconsider structuring new matrices according to their new matrices according to their new judgements and following this procedure until the matrices with satisfied CR are too long period of work needed.

Thus for given comparison matrix, If CR ≥ 0,1. We can adopt some techniques to adjust the comparison matrix properly such that the revised matrix possesses acceptable consistency (CR<0,1). Then, from the revised matrix, we can derive the reasonable priority vector of the original one by the eigenvector method.

2. Methodology

Let

$$R_n^+ = \{x = (x_1, x_2, \ldots, x_n)^T \mid x_i > 0, i = 1,2,\ldots, n\}$$

Lemma 2.1 . Let A=(a_{ij}) be an n positive matrix and λ_{max} be the maximal eigenvalue of A. Then

$$\lambda_{max} = \min_{x \in R_n^+} \max \sum_{j=1}^{n} a_{ij} \frac{x_j}{x_i}$$

Let A and λ_{max} be as in Lemma 2.1. The positive rigth eigenvector corresponding to λ_{max} is called the principal right eigenvector of A.

Lemma 2.2 Let x>0, y>0, λ>0 and μ>0 , and λ+μ=1. Then

$$x^{\lambda} y^{\mu} \leq \lambda x + \mu y$$

with equality if and only if x=y

Lemma2.3 Let A be an n positive reciprocal matrix, λ_{max} be the maximal eigenvalue of A. Then $\lambda_{max} \geq n$ and equality holds if and only if A is consistent.

Theorem2.1. Let $A=(a_{ij})$ be an n positive reciprocal matrix, λ_{max} be the maximal eigenvalue of A and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ be the principal right eigenvector of A. Let $B=(b_{ij})$, where

$$b_{ij} = (a_{ij})^{\lambda} \cdot \left(\frac{w_i}{w_j} \right)^{1-\lambda}$$

Let μ_{max} be the maximal eigenvalue of B. Then $\mu_{max} \leq \lambda_{max}$, with equality if and only if A is consistent.

Proof. Let

$$e_{ij} = a_{ij}(\omega j / \omega i), i, j = 1,2,...n.$$

Then

$$\lambda \max = \sum_{j=1}^{n} e_{ij}$$

and

$$b_{ij} = e_{ij}^{\lambda} \cdot \left(\frac{w_j}{w_i} \right)$$

from Lemmas 2.1-2.3, we have

$$\mu \max = \min_{x \in R_n^+} \max_i \sum_{j=1}^{n} b_{ij} \frac{x_j}{x_i} \leq \max_i \sum_{j=1}^{n} b_{ij} \frac{w_j}{w_i}$$

$$= \max_i \sum_{j=1}^{n} e_{ij}^{\lambda} \leq \max_i \sum_{j=1}^{n} (\lambda e_{ij} + 1 - \lambda) \leq \lambda \lambda_{\max} + (1-\lambda) n \leq \lambda_{\max}$$

with equality if and only if $\lambda_{max}=n$, that is, A is consistent.

In order to prove the convergence of above theory
Let $A= (a_{ij})$ be an inconsistent comparison matrix. Let $A^{(k)}$ be the matrix sequence generated and λ_{max} be the maximal eigenvalue of $A^{(k)}$. Then for each k,

$$\lambda \max(A^{(k+1)}) < \lambda \max(A^{(k)})$$

and

$$\lim_{k \to \infty} \lambda \max(A^{(k)}) = n$$

Proof. Since $CR^{(k)}>0$, $A^{(k)}$ is inconsistent. By Theorem 2.1, we have

$$\lambda \max(A^{(k+1)}) < \lambda \max(A^{(k)})$$

For each k.
Two criteria of modificatory effectiveness are proposed as follows.

$$\delta = \max_{ij} \{| a_{ij}^{m} - a_{ij}^{0} |\}, \quad i, j = 1, 2, \ldots n$$

and

$$\sigma = \frac{\sqrt{\sum_{j=1}^{n} \sum_{i=1}^{n} (a_{ij}^{(m)} - a_{ij}^{(0)})^2}}{n}$$

For a comparison matrix structured on one to nine scale, if $\delta<2$ and $\sigma<1$, the modification can be regarded as acceptable.

From the above theory, following step by step to modify the comparison matrix.

Let $A=(a_{ij})$ be an n comparison matrix, k be the number of iterative times and $0<\lambda<1$
Step 1 : Let $A^{(0)}=(a_{ij}^{(0)})= (a_{ij})$, CR = 0.1 and k=0.

Step 2 : Calculate the maximal eigenvalue $\lambda_{max}(A^{(k)})$ of $A^{(k)}$ and the normalised principal right eigenvector
$\omega^{(k)} = (\omega_1^{(k)}, \omega_2^{(k)},\ldots, \omega_n^{(k)})^T$
Step 3 : Calculate the consistency index, $CI^{(k)} = (\lambda_{max}(A^{(k)} - n)/(n-1)$ and the consistency ratio $CR^{(k)} = CI^{(k)}/RI$
Step 4 : If $CR^{(k)} < CR$, then go to step 6; otherwise, continue the next step.
Step 5 : Let $A^{k+1} = (a_{ij}^{(k+1)})$, where

$$a_{ij}^{(k+1)} = (a_{ij}^{(k)})^{\lambda} \cdot \left(\frac{w_i^{(k)}}{w_j^{(k)}} \right)^{1-\lambda}$$

Let k = k + 1 and return to step 2.
Step 6 : Calculate δ and σ, If $CR<0.1$, $\delta<2$ and $\sigma<1$ then
Step 7 : Output k, $A^{(k)}$, $(\lambda_{max}(A^{(k)})$, $CR^{(k)}$, $w^{(k)}$ and $A^{(k)}$ is the modified positive reciprocal matrix and $w^{(k)}$ is the vector of priorities.
Step 8 : End.

To compare the result from resurvey and improving consistency method is used Kendall rank correlation coefficient. The Kendall rank correlation coefficient, τ (tau) is suitable as a measure of correlation with the same sort of data for which τ_s is useful. That is, if at least ordinal measurement of both the X and Y variables has been achieved, so that every subject can be assigned a rank on both X and Y, then τ will give a measure of the degree of association or correlation between the two sets of ranks.

Suppose we ask judge X and judge Y to rank four objects. For example, we might ask them to rank four essays in order of quality of expository style. We represent the four papers as a, b c, and d. The obtained rankings are these:

Criteria	A	B	C	D
Resurvey	3	4	2	1
Method	3	1	4	2

If we rearrange the order of the essays so that judge X's ranks appear in natural order, we get

Criteria	D	C	A	B
Resurvey	1	2	3	4
Method	2	4	3	1

We are now in a position to determine the degree of correspondence between the judgements of X and Y. Judge X's rankings being in their natural order, we proceed to determine how many pairs of ranks in judge Y's set are in their correct (natural) order with respect to each other

Consider first all possible pairs of ranks in which judge Y's rank 2 the rank farthest to the left in his set, is one member. The first pair 2 and 4, has the correct order : 2 precedes 4. Since the order is natural, we assign a score of +1 to this pair. Rank 2 and 3 constitute the second pair. This pair is also in the correct order, so it also earns a score of +1. Now the third pair consists of ranks 2 and 1. These ranks are not in natural order; 2 precedes 1. Therefore we assign this pair a score of –1. For all pairs which include the rank 2, we total the scores: (+1) + (+1) + (-1) = +1

Now we consider all possible pairs of ranks which include rank 1(which) is the rank second from the left in judge Y's set) and one succeeding rank. One pair is 4 and 3; the two members of the pair are not in the natural order, so the score for that pair is –1. Another pair is 4 and 1; again a score of –1 is assign. The total of these scores is (-1) + (-1) = -2

When we consider rank 3 and succeeding ranks, we get only this pair : 3 and 1. The two members of this pair are in the wrong order; therefore this pair receives a score of –1.

The total of all the scores we have assigned is (+1) + (-2) + (-1) = -2

Now that is the maximum possible total we could have obtained for the score assigned all the pairs in judge Y's ranking. The maximum possible total would have been yielded if the rankings of judges X and Y had agreed perfectly, for then, when the rankings of judge X were arranged in their natural order, every pair of judge Y's ranks would also be in the correct order and thus every pair would receive a score of +1. The maximum possible total then, the one which would occur in the case of perfect agreement between X and Y, would be four things taken two at a time = 6

The degree of relation between the two sets of ranks is indicated by the ratio of the actual total of +1's and –1's to the possible maximum total. The Kendall rank correlation coefficient is that ratio:

$$\tau = -\tfrac{2}{6} = -0{,}33$$

That is, τ = -0.33 is a measure of the agreement between the ranks assigned to essays by judge X and those assigned by judge Y.

One may think of τ as a function of the minimum number of inversions or interchanges between neighbours which is required to transform one ranking into another. That is, τ is sort of coefficient of disarray.

Method

We have seen that

$$\tau = \frac{actualscore}{\max imumpossiblescore}$$

In general, the maximum possible score can expressed 1/2N(N-1). Where N= the number of objects or individuals ranked on both X and Y.

$$\tau = \frac{S}{\frac{1}{2}N(N-1)}$$

If a random sample is drawn from some population in which X and Y are unrelated, and the members of the sample are ranked on X and Y, then for any given order of the X ranks all possible orders of Y ranks are equally likely. Table Q may be used to determine the exact probability associated with the occurrence (one tailed) under Ho of any value as extreme as an observed S.

In the research, we use three models and seven respondents to implement this journal into reality. They are Employee selection, Car selection, and Journal selection. And from all that part(three models and seven respondents) we got 123 matrices.

3. Application

Inconsistency matrix is shown with CR=0,178

$$\begin{bmatrix} 1 & \frac{1}{7} & \frac{1}{2} & 5 & \frac{1}{9} \\ 7 & 1 & 1 & 7 & 1 \\ 2 & 1 & 1 & 5 & 3 \\ \frac{1}{5} & \frac{1}{7} & \frac{1}{5} & 1 & \frac{1}{9} \\ 9 & 1 & \frac{1}{3} & 9 & 1 \end{bmatrix}$$

and then by resurveyed we get revised matrix with CR = 0,0926

$$\begin{bmatrix} 1 & \frac{1}{6} & \frac{1}{2} & 5 & \frac{1}{7} \\ 6 & 1 & 1 & 7 & 1 \\ 2 & 1 & 1 & 6 & 2 \\ \frac{1}{5} & \frac{1}{7} & \frac{1}{6} & 1 & \frac{1}{9} \\ 7 & 1 & \frac{1}{2} & 9 & 1 \end{bmatrix}$$

and from the improving consistency method we get revised matrix with CR =0,0976, δ= 1,7768 and σ=0,3593

$$\begin{bmatrix} 1 & 0,16 & 0,43 & 4,12 & 0,4 \\ 6,16 & 1 & 1,03 & 7,55 & 1,03 \\ 2,34 & 0,98 & 1 & 5,72 & 2,3 \\ 0,24 & 0,13 & 0,18 & 1 & 0,11 \\ 7,22 & 0,97 & 0,44 & 8,86 & 1 \end{bmatrix}$$

By using Kendall Tau's procedure, we arrange the the derived weight into rank

Criteria	Weight from resurvey	Rank	Weight from Improving consistency method	Rank
A	0,0767	4	0,0661	4
B	0,3184	1	0,5554	1
C	0,2887	2	0,0904	3
D	0,0335	5	0,032	5
E	0,2827	3	0,2561	2

and rearrange the rank the order of the criteria so that resurvey's ranks appear in natural order

Criteria	B	E	C	A	D
Resurvey's rank	1	2	3	4	5
Improving consistency method's rank	1	3	2	4	5

and we get the result

Code	S	τ	P	α	Uji Ho
Value	8	0,8	0,042	0,05	Reject Ho

That is mean correlation between resurvey and improving consistency method are equally likely

4. Conclusion

There are some weaknesses by using improving consistency method;

1. This method is not perfect enough, because there are some numeric result which are out of 9 scale.
2. From the experiment 32 inconsistency only 15 matrices can be revised, around 46,88%. And only 66,67%of them are significantly identic.
3. Improving consistency method tends to revise the comparison matrix with the same or almost same weight from inconsistency matrix. On the other hand, this method cannot revise the wrong derived priorities. which mean's, it's not accurate.

There are some weakness in application improving consistency method, although this method practicable in short time, but can not achieve the truth by using resurvey. The suggestion is using the resurvey to revise the inconsistency is still the best solution.

References

[1] Saaty, Thomas.L. The analytic hierarchy process, McGraw-Hill, New York,1980.

[2] Saaty, Thomas,L. & Vargas,L. Inconsistency and rank preservation. Journal of mathematical psychology1984,28.205-214

[3] Saaty, Thomas.L. Multicriteria Decision Making- The analityc Hierarchy Process, Vol. 1 of the AHP series, RWS Publications, Pittsburgh, 1992.
[4] Xu Zeshui and Wei Cuiping. A consistency improving method in the analytic hierarchy process, European Journal Of Operational Research 116 (1999) 443-449.
[5] Siegel, Sidney. Non Parametric Statistics, McGraw-Hill,1956